On the embeddings of quasi-categories into prederivators

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Overview

Why higher categories?

- homotopy coherent structures arising in (stable) homotopy theory, (derived) algebraic geometry and so on,
- natural evolution of category theory.

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- homotopy coherent structures arising in (stable) homotopy theory, (derived) algebraic geometry and so on,
- natural evolution of category theory.

An important problem

Technical difficulties involved in the definition and application of higher categories lead to a proliferation of models of these structures.

Theorem (*Toën*, 2005)

All models of $(\infty, 1)$-categories define fibrant objects of Quillen equivalent model categories.

The general motivation

Introduce a theoretical framework in which one can study all these models at once, i.e. work "model independently".

State of the art of Synthetic Higher Category Theory

- Model-independent Higher Category Theory via ∞-cosmoi (Dominic Verity and Emily Riehl)
- HoTT as an internal logic for ∞-topoi (Shulman, Kapulkin, Lumsdaine, Riehl etc.)

The purpose of the thesis

Give a concrete example of the first philosophy, by analysing the interactions between two of these models.

Simplicial sets

Let $\mathbf{\Delta}$ be the category of finite ($\neq \emptyset$) ordinals $[n] = (0 < \cdots < n)$ and order preserving maps (*simplex category*).

Lemma

All the morphisms in $\mathbf{\Delta}$ can be *uniquely* written as compositions of *cofaces* and *codegeneracies*, which are defined as follows.

$$d^k \colon [n-1] \to [n] \qquad j \longmapsto \begin{cases} j, & j < k \\ j+1, & j \geq k \end{cases}$$

$$s^k \colon [n+1] \to [n] \qquad j \longmapsto \begin{cases} j, & j \leq k \\ j-1, & j > k \end{cases}$$

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Definition

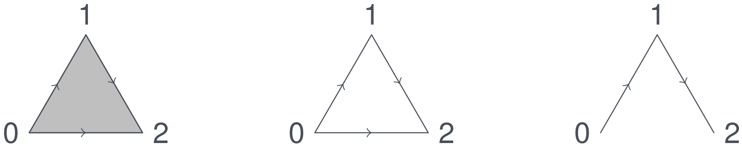
A *simplicial set* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$.

The presheaf category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ is denoted by **sSet**. Furthermore $d_k := X(d^k),\ s_k := X(s^k),\ X_n := X([n])$ are called resp. *faces*, *degeneracies* and *the set of n-simplices*, for every $X \in \mathbf{sSet}$.

Main examples

1. the standard n-simplex $\Delta^n := \mathbf{\Delta}(-, [n])$,
2. its boundary $\partial\Delta^n$,
3. the k^{th} horn Λ^n_k, for $0 \leq k \leq n$.

For instance, if $n = 2$ we have (amongst others) these ones.



Δ^2 $\quad$ $\partial\Delta^2$ $\quad$ Λ^2_1

Definition

A simplicial set X is a *Kan complex* if every horn $\Lambda^n_k \to X$, for $0 \leq k \leq n$, has a filler $\Delta^n \to X$, i.e.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

commutes.

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commutes.

Example: the singular complex of a space

$\mathrm{Sing}(X)\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set},\ [n] \mapsto \mathbf{Top}(|\Delta^n|, X)$ is a Kan complex.

The nerve of a category

Definition

The nerve is defined as $N\colon \mathbf{Cat} \to \mathbf{sSet},\ \mathbf{C} \mapsto \mathbf{Cat}(-, \mathbf{C})$.

I.e. n-simplices of $N\mathbf{C}$ are strings of composable morphisms

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \to \cdots \xrightarrow{f_n} C_n,$$

faces compose morphisms and degeneracies add identities.

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Example

The nerve of a groupoid is a Kan complex (with unique fillers).

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Example

The nerve of a groupoid is a Kan complex (with unique fillers).

Proposition

The nerve functor is fully faithful.

Quasi-categories

Definition

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Examples

- every Kan complex is a quasi-category,
- the nerve of a category is a quasi-category (with unique fillers).

Remark

Quasi-categories generalize both Kan complexes and categories, connecting homotopy theory and category theory.

We denote by **qCat** the full subcategory of **sSet** spanned by quasi-categories.

Proposition

If X is a simplicial set and Y is a quasi-category, then Y^X is a quasi-category.

Definition

$\mathbf{qCat}_\bullet$ is the simplicially enriched category such that, for every pair of quasi-categories X, Y, the simplicial set of morphisms is

$$\begin{aligned}
\mathbf{qCat}_\bullet(X, Y)\colon \boldsymbol{\Delta}^{\mathrm{op}} &\to \mathbf{Set} \\
[n] &\mapsto \mathbf{qCat}(X, Y^{\Delta^n}) \\
([m] \to [n]) &\mapsto \mathbf{qCat}(X, Y^{\Delta^n}) \to \mathbf{qCat}(X, Y^{\Delta^m})
\end{aligned}$$

Homotopies inside a quasi-category

Definition

Two 1-simplices $f, g\colon x \to y$ in a quasi-category X are said to be *homotopic* if there exists a 2-simplex $\sigma\colon \Delta^2 \to X$ such that $d_2(\sigma) = \mathrm{id}_x, d_0(\sigma) = g$ and $d_1(\sigma) = f$.

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

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\end{tikzcd}$$

Remark

The homotopy category $\mathrm{Ho}(X)$ of a quasi-category X is defined by passing to homotopy classes of 1-simplices.

Proposition

The functor $\mathrm{Ho}\colon \mathbf{qCat} \to \mathbf{Cat}$, which sends a quasi-category to its homotopy category, is left adjoint to N.

Remark

The adjunction $\mathrm{Ho} \dashv N$ is an instance of the general *nerve-realization* adjunction.

Definition

$\underline{\mathbf{qCat}}$ is the 2-category having quasi-categories as 0-cells and such that, for every pair of quasi-categories X and Y, the hom-category is defined to be $\underline{\mathbf{qCat}}(X, Y) := \mathrm{Ho}(Y^X)$.

Remark

A model structure is *uniquely* determined by any of the following information:

(i) the cofibrations and weak equivalences,
(ii) the fibrations and weak equivalences,
(iii) the cofibrations and fibrations,
(iv) the cofibrations and fibrant objects,
(v) the fibrations and cofibrant objects.

Joyal's model structure

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Proposition

There exists a model structure on **sSet** s.t. the cofibrations are the monomorphisms and the fibrant objects are the quasi-categories. This is called the *Joyal model structure*.

Prederivators

Definition

A *prederivator* is a strict 2-functor $\mathbb{D} \colon \underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$.

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Example

Let $\mathbf{C}$ be a category. Then $\mathrm{y}_{\mathbf{C}}\colon \underline{\mathbf{Cat}}^{\mathrm{op}} \to \underline{\mathbf{CAT}},\ J \mapsto \mathbf{C}^J$ is called the *prederivator represented by* $\mathbf{C}$.

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Example

Let $\mathbb{D}$ be a prederivator and let M be a fixed category. Then

$$\begin{aligned} \mathbb{D}^M\colon \underline{\mathbf{Cat}}^{\mathrm{op}} &\to \underline{\mathbf{CAT}} \\ J &\mapsto \mathbb{D}^M(J) = \mathbb{D}(M \times J) \end{aligned}$$

is called the *shifted prederivator*.

Example

If X is a quasi-category we define the prederivator associated to X as follows

$$\begin{aligned} \mathbf{Ho}_X \colon \underline{\mathbf{Cat}}^{\mathrm{op}} &\to \underline{\mathbf{CAT}} \\ J &\mapsto \mathrm{Ho}(X^{N(J)}) \end{aligned}$$

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Definition

A *morphism of prederivators* $F \colon \mathbb{D} \to \mathbb{D}'$ is a pseudonatural transformation between them.

Definition

A *natural transformation* $\tau \colon F \Rightarrow G$ is a modification of the pseudonatural transformations F and G.

Enhancements of prederivators

Definition

$\mathbf{PDer}_\bullet$ is the simplicially enriched category such that, for every pair of prederivators $\mathbb{D}_1, \mathbb{D}_2$, the simplicial set of morphisms is

$$\begin{aligned}
\mathbf{PDer}(\mathbb{D}_1, \mathbb{D}_2) \colon \mathbf{\Delta}^{\mathrm{op}} &\to \mathbf{Set} \\
[n] &\mapsto \mathbf{PDer}^{\mathrm{st}}(\mathbb{D}_1, \mathbb{D}_2^{[n]}) \\
([m] \to [n]) &\mapsto \mathbf{PDer}^{\mathrm{st}}(\mathbb{D}_1, \mathbb{D}_2^{[n]}) \to \mathbf{PDer}^{\mathrm{st}}(\mathbb{D}_1, \mathbb{D}_2^{[m]})
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Definition

$\underline{\mathbf{PDer}}$ is the 2-category having prederivators as 0-cells, morphisms of prederivators as 1-cells and natural transformations between those morphisms as 2-cells.

Simplicial embedding

Definition

Let us define the functor

$$\begin{aligned} \mathbf{Ho}\colon \mathbf{qCat} &\to \mathbf{PDer}^{\mathrm{st}} \\ X \quad &\mapsto \mathbf{Ho}_X \\ X \to Y &\mapsto \mathbf{Ho}_X \to \mathbf{Ho}_Y \end{aligned}$$

where the action on morphisms is obtained componentwise by applying the functor $(-)^{N(J)}$ and then the functor Ho, for every $J \in \mathbf{Cat}$.

Proposition

$\mathbf{Ho}$ extends to a simplicial functor $\mathbf{Ho}_\bullet \colon \mathbf{qCat}_\bullet \to \mathbf{PDer}_\bullet$.

Sketch of the proof

- the action on objects is given by that of **Ho**,
- consider the action of **Ho** between the set of morphisms of **qCat** and **PDer**: $\mathbf{qCat}(X, Y^{\Delta^n}) \xrightarrow{\mathbf{Ho}} \mathbf{PDer}(\mathbf{Ho}_X, \mathbf{Ho}_{Y^{\Delta^n}})$,
- notice that $\mathbf{qCat}(X, Y^{\Delta^n}) = \mathbf{qCat}_n(X, Y)$ by definition and check that $\mathbf{PDer}(\mathbf{Ho}_X, \mathbf{Ho}_{Y^{\Delta^n}}) \cong \mathbf{PDer}(\mathbf{Ho}_X, \mathbf{Ho}_Y^{[n]}) = \mathbf{PDer}_n(\mathbf{Ho}_X, \mathbf{Ho}_Y)$ via a chain of isomorphisms,
- define $\mathbf{Ho}_\bullet$ so that on n-simplices it agrees with **Ho**.

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- define $\mathbf{Ho}_\bullet$ so that on n-simplices it agrees with **Ho**.

Theorem (*Carlson*, 2016)

$\mathbf{Ho}_\bullet$ is simplicially fully faithful.

Sketch of the proof

- prove that **Ho** is fully faithful (not so easy),
- extend it to $\mathbf{Ho}_\bullet$ (easy).

2-categorical embedding

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Proposition

Ho extends to a 2-functor $\underline{\mathbf{Ho}}\colon \underline{\mathbf{qCat}} \to \underline{\mathbf{PDer}}$.

Sketch of the proof

- define $\underline{\mathbf{Ho}}$ just like **Ho** at the level of 0-cells,
- define a 2-categorical enhancement of $\mathrm{Ho}\colon \mathbf{qCat} \to \mathbf{Cat}$ via the product-hom adjunction of **qCat**,
- define a 2-categorical enhancement of the nerve functor using the isomorphism $N(\mathbf{B}^{\mathbf{A}}) \cong N(\mathbf{B})^{N(\mathbf{A})}$.

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Theorem (*Carlson*, 2016)

The restriction of the 2-functor $\underline{\mathbf{Ho}}$ to the 2-category of small quasi-categories is bicategorically fully faithful.

Model structure on prederivators

Recall that $x \in X_n$ is *degenerate* if $x = s_i y$, for some $y \in X_{n-1}$ and $0 \leq i \leq n$, otherwise it is *nondegenerate*.

Definition

A category is called *homotopy finite* if its nerve has finitely many nondegenerate simplices.

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Remarks

1. $[n]$ is homotopy finite, for every $n \in \mathbb{N}$.
2. Homotopy finite $\Rightarrow$ finite, but the converse is not true (take a finite group G seen as a one-object groupoid with G as its finite group of automorphisms).
3. We use **HFin** to denote the 2-category of homotopy finite categories.

$\mathbf{PDer}^{\mathrm{st}}_{\mathrm{HFin}}$ is the category of 2-functors $\underline{\mathbf{HFin}}^{\mathrm{op}} \to \underline{\mathbf{Cat}}$ and strict natural transformations.

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Theorem (*Fuentes-Keuthan, Kedziorek and Rovelli*, 2018)

There exists a model structure on $\mathbf{PDer}^{st}_{\mathrm{HFin}}$ that is Quillen equivalent to the Joyal model structure on **sSet**.

Sketch of the proof

- construct a pair of adjoint functors $\begin{tikzcd} \mathbf{sSet} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{PDer}^{st}_{\mathrm{HFin}} \arrow[l, bend left, "R"] \end{tikzcd}$,
- F is a w.e./fib. in $\mathbf{PDer}^{st}_{\mathrm{HFin}} \overset{def}{\Longleftrightarrow}$ RF is a Joyal w.e./fib.,
- $L \dashv R$ is a Quillen adjunction thanks to a Kan's theorem,
- $L \dashv R$ is a Quillen equivalence (easy check).