Monads and their applications II

Dr. Daniel Schäppi's course lecture notes

by

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2-Monads and Their 2-Categories of Algebras

0.1 Introduction

These notes will focus on 2-dimensional monad theory, which can be viewed as the study of algebraic structures on 2-categories. Like in the one-dimensional case, after defining a 2-monad we concern ourselves with the categories of algebras it defines, however the higher dimension allows to relax the definitions and observe how different coherence conditions lead to different (and generally less well-behaved) objects.

One may ask why we are keen to better understand 2-monads. One answer is that, similarly to the 1-dimensional case, this allows us to better understand other 2-categories, perhaps with additional structure (i.e. monoidal, braided, some kinds of limits, etc) by relating them to 2-categories of algebras.

We now start recalling some relevant definitions and facts which we will need later on.

In order to carry out our project we shall work with $\mathcal{V}$-cosmos and presentability conditions.

Definition 0.1.1. A cosmos $\mathcal{V}$ is a complete, cocomplete symmetric monoidal closed category.

Definition 0.1.2. An object c in a $\mathcal{V}$-category $\mathcal{C}$ is κ-presentable if $\mathcal{C}(c,-)\colon \mathcal{C} \to \mathcal{V}$ preserves κ-filtered colimits. This is equivalent to saying that the functor $\mathcal{C}(c,-)\colon \mathcal{C}_0 \to \mathcal{V}_0$ is κ-accessible, where $\mathcal{C}_0$ and $\mathcal{V}_0$ are the underlying categories.

Theorem 0.1.3. Let $\mathcal{V}$ be a lfp cosmos. Then $\mathcal{V}$-**Cat** is a lfp cosmos and a lfp 2-category.

By studying monads in this setting we achieve a great level of generality since our results will not depend on the underlying enrichment, thus unifying many contexts.

But what is a 2-monad?

Definition 0.1.4. A 2-monad is a monad in the 2-category 2-**CAT** of locally small 2-categories, 2-functors and (strict) 2-natural transformations.

We will often construct them using presentations, that is via colimit constructions and free 2-monads on 2-endofunctors. This is achieved through the following results.

Theorem 0.1.5. Let $\mathcal{V}$ be a lfp cosmos, $\mathcal{C}$ a locally κ-presentable $\mathcal{V}$-category. Then the forgetful functor

$$\mathcal{V} - \mathsf{Mnd}_\kappa(\mathcal{C}) \to \mathcal{V} - \mathbf{CAT}_\kappa(\mathcal{C},\mathcal{C})$$

is monadic. Moreover, it preserves colimits.

Corollary 0.1.6. In the above situation, the functor

$$(-)\text{-}\mathsf{Alg}\colon \mathcal{V} - \mathsf{Mnd}_\kappa(\mathcal{C}) \to \mathcal{V} - \mathbf{CAT}/\mathcal{C}$$

sends colimits to limits.

Remark 0.1.7. In general, $\mathcal{V} - \mathsf{Mnd}_\kappa(\mathcal{C})$ is not a $\mathcal{V}$-category. This is because monads are monoids in a monoidal $\mathcal{V}$-category of endofunctors, but monoids in general do not define a $\mathcal{V}$-category: for example, consider $\mathsf{Mon}(\mathbf{Ab}) = \mathbf{Ring}$, which is not even additive.

This has to do with the non-existence of a "diagonal" $\mathcal{V}$-functor $\mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$. In particular, if $\mathcal{V}$ is cartesian then this problem does not arise and indeed for $\mathcal{V} = \mathbf{Cat}$ we expect the monadic adjunction 0.1.5 to be enriched.

Unfortunately, we can't apply the theorem above to show the corollary. Instead, we use it to give a presentation of a 2-monad whose algebras are 2-monads with rank κ.

Given a monoidal 2-category $\mathcal{M}$ (i.e. the associator $(A \otimes B) \otimes C \to A \otimes (B \otimes C)$ is 2-natural, satisfies the pentagon axioms, etc), we have a 2-category $\mathsf{Mon}(\mathcal{M})$ of monoids $(M, \mu\colon M \otimes M \to M, \eta\colon I \to M)$ in $\mathcal{M}$ with 1-cells the monoid morphisms and 2-cells the 2-cells $\alpha\colon f \Rightarrow g\colon M \to N$ in $\mathcal{M}$ s.t.

$$\begin{tikzcd} M \otimes M \arrow[r, "\mu_M"'] & M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & N \end{tikzcd} \quad = \quad \begin{tikzcd} M \otimes M \arrow[r, bend left, "f\otimes f"] \arrow[r, bend right, "g\otimes g"'] \arrow[r, phantom, "\Downarrow\alpha\otimes\alpha"] & N \otimes N \arrow[r, "\mu_N"'] & N \end{tikzcd},$$

$$\begin{tikzcd} I \arrow[r, "\eta_M"'] & M \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & N \end{tikzcd} \quad = \quad \mathrm{id}_{\eta_N}$$

hold.

If $- \otimes -$ preserves κ-filtered colimits in each variable, then the 2-functors $FM = M \otimes M$, $GM = (M \otimes M) \otimes M + M + M$ are κ-accessible and we have two natural ways to go from F-algebras to G-algebras.

The coequalizer of the resulting pair of maps on the free monads $TG \rightrightarrows TF$ gives us a presentation of a 2-monad T as a coequalizer. It has $T\text{-}\mathsf{Alg} \cong \mathsf{Mon}(\mathcal{M})$ by construction if $\mathcal{M}$ is locally κ-presentable as a 2-category.

Let $\mathcal{K}$ be a locally κ-presentable 2-category, i.e. $\mathcal{V} - \mathbf{Cat}$ and specifically $\mathbf{Cat}$, and let $\mathcal{M} = [\mathcal{K}, \mathcal{K}]_\kappa$. Then the category of κ-accessible endofunctors on $\mathcal{K}$, that is $\mathcal{M}$, is itself locally κ-presentable.

Notice that the composition preserves κ-filtered colimits in each varible. Indeed, for F^* it's clear and for F_* is too since F is κ-accessible.

Monoids in $\mathcal{M}$ are 2-monads!

To show that $2-\mathsf{Mnd}_\kappa(\mathcal{K}) \to 2-\mathsf{Mnd}(\mathcal{K})$ preserves colimits we need the following proposition.

Proposition 0.1.8. Let F be a strong monoidal 2-adjoint $\mathcal{M} \to \mathcal{M}'$. Then the right 2-adjoint inherits a lax monoidal structure s.t. unit and counit are monoidal. Both 2-functors lift to the 2-categories of monoids, so $\mathsf{Mon}(F)\colon \mathsf{Mon}(\mathcal{M}) \to \mathsf{Mon}(\mathcal{M}')$ is a left 2-adjoint.

Proof. Exercise. □

We can now prove what we stated earlier.

Theorem 0.1.9. Let $\mathcal{K}$ be a locally κ-presentable 2-category. Then the forgetful 2-functor

$$2 - \mathsf{Mnd}_\kappa(\mathcal{K}) \to [\mathcal{K}, \mathcal{K}]_\kappa$$

is 2-monadic and κ-accessible. In particular, $2 - \mathsf{Mnd}_\kappa(\mathcal{K})$ is a locally κ-presentable 2-category. Moreover, the inclusion

$$2 - \mathsf{Mnd}_\kappa(\mathcal{K}) \to 2 - \mathsf{Mnd}(\mathcal{K})$$

preserves colimits and in fact it is a left adjoint.

Proof. We have $2-\mathsf{Mnd}_\kappa(\mathcal{K}) = \mathsf{Mon}([\mathcal{K},\mathcal{K}]_\kappa)$, so the above discussion shows that there is a κ-accessible 2-monad on $[\mathcal{K},\mathcal{K}]_\kappa$ with $T\text{-}\mathsf{Alg} \cong 2-\mathsf{Mnd}_\kappa(\mathcal{K})$.

For the second part, recall that left Kan extensions along the inclusion $J\colon \mathcal{K}_\kappa \to \mathcal{K}$ of κ-presentable objects gives an equivalence of 2-categories $[\mathcal{K}_\kappa,\mathcal{K}] \to [\mathcal{K},\mathcal{K}]_\kappa$ (this is true for a general lfp cosmos —-missing bit, it was 11:23—-).

It follows that the inclusion $[\mathcal{K},\mathcal{K}]_\kappa \to [\mathcal{K},\mathcal{K}]$ is, up to equivalence, given by the left Kan extension along J. (Check and finish this proof) ☐

This will allows us to write presentations of 2-monads for 2-categories such as $\mathbb{R}$-linear categories, simplicial categories, etc, which has two important consequences: firstly, when constructing a 2-monad from free monads we may also use weighted colimits; secondly, since 2-monads with rank κ are algebras for a 2-monad with rank κ, any general theorem we prove about algebras gives a corresponding 2-monad with rank κ.

As we mentioned earlier, we may be interested in less strict definitions compared to the 1-dimensional case. Here we start considering them by specifying new classes of morphisms of algebras.

Definition 0.1.10. Let T be a 2-monad, (A,a), (B,b) two T-algebras.

A lax T-morphism is a pair $(f,\overline{f})$ where $f\colon A\to B$ is a 1-cell and $\overline{f}\colon b\cdot Tf \to f\cdot a$ is a 2-cell such that the equations

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T^2f"'] \arrow[dr, phantom, "="] & TA \arrow[r, "a"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overline{f}\,\Nearrow"] & A \arrow[d, "f"] \\ T^2B \arrow[r, "\mu_B"'] & TB \arrow[r, "b"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "T^2f"'] \arrow[dr, phantom, "T\overline{f}\,\Nearrow"] & TA \arrow[r, "a"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overline{f}\,\Nearrow"] & A \arrow[d, "f"] \\ T^2B \arrow[r, "Tb"'] & TB \arrow[r, "b"'] & B \end{tikzcd},$$

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & TA \arrow[r, "a"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overline{f}\,\Nearrow"] & A \arrow[d, "f"] \\ B \arrow[r, "\eta_B"'] & TB \arrow[r, "b"'] & B \end{tikzcd} \quad = \quad \mathrm{id}_f$$

hold.

A lax T-morphism is a pseudo T-morphism if $\overline{f}$ is an isomorphism and it is strict if $\overline{f} = \mathrm{id}$.

A colax or oplax T-morphism is a lax T-morphism with the direction of $\overline{f}$ reversed and the equations adapted.

A 2-cell between lax/pseudo/strict T-morphisms $\alpha\colon (f,\overline{f}) \Rightarrow (g,\overline{g})$ is a 2-cell $\alpha\colon f\Rightarrow g$ s.t.

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"', bend right] \arrow[dr, phantom, "\overset{\overline{f}}{\Longrightarrow}"] & A \arrow[d, "f"', bend right] \arrow[d, "g", bend left] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ TB \arrow[r, "b"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} TA \arrow[r, "a"] \arrow[d, bend right] \arrow[d, bend left] \arrow[d, phantom, "\overset{T\alpha}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\overline{g}}{\Longrightarrow}"] & A \arrow[d, "g", bend left] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

We write $T\text{-}\mathsf{Alg}_S$, $T\text{-}\mathsf{Alg}_P$ and $T\text{-}\mathsf{Alg}_L$ for the 2-categories of T-algebras, strict/pseudo/lax T-morphisms and 2-cells as above.

(Other missing bit)

0.2 Presentations of 2-Monads

We have defined two 2-categories $T\text{-}\mathsf{Alg}_P$, $T\text{-}\mathsf{Alg}_L$ of pseudo and lax morphisms respectively for a 2-monad T. We want to understand how to describe them when T is given by a presentation.

We remember that in a complete 2-category $\mathcal{K}$ we have a 2-endofunctor $< A, B >: \mathcal{K} \to \mathcal{K}$ for each pair of objects A, B in $\mathcal{K}$ given by the right Kan extension of $B: * \to \mathcal{K}$ along $A: * \to \mathcal{K}$. In particular, $< A, B > C = B^{\mathcal{K}(C,A)}$ and, if $A = B$, this defines a 2-monad, just like in the 1-dimensional case. Moreover, the 2-monad morphisms $T \Rightarrow < A, B >$ are in natural bijection with T-algebra structures on A.

Now we can form for any pair of 1-cells $f, g: A \to B$ in $\mathcal{K}$ the (iso???) comma object

$$\begin{tikzcd} \{f,g\}_{p/l} \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\swarrow\lambda"] & < A,A > \arrow[d, "<A,f>"] \\ < B,B > \arrow[r, "<g,B>"'] & < A,B > \end{tikzcd}$$

in $[\mathcal{K}, \mathcal{K}]$. If $f = g$, then this is again a 2-monad and 2-monad morphisms $T \to \{f, f\}_{p/l}$ correspond to (pseudo) lax T-morphism structures on the 1-cell f. More precisely, such a morphism corresponds to a T-algebra structure on A and one on B, namely $c \cdot \gamma$ and $d \cdot \gamma$ and a (invertible) 2-cell $\overline{f}: Tf \cdot b \Rightarrow f \cdot a$ corresponding to $\lambda \cdot \gamma$ s.t. $(f, \overline{f})$ is a lax (pseudo) T-morphism.

$$\begin{tikzcd} {[\rho,\rho]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{f,f\} \arrow[d, "\{f,\rho\}_l"] \\ \{g,g\}_l \arrow[r, "\{\rho,g\}_l"'] & \{f,g\}_l \end{tikzcd}$$

which inherits a 2-monad structure for which a 2-monad morphism $T \Rightarrow [\rho, \rho]$ exists if and only if ρ is a T-transformation between $(f, \overline{f})$ and $(g, \overline{g})$.

These facts can be used to identify $T\text{-}\mathsf{Alg}_P$ and $T\text{-}\mathsf{Alg}_S$ is T is given as a (weighted) colimit of free monads.

Example 0.2.1. Let's consider the 2-monad of monads in a monoidal 2-category $\mathcal{M}$ as above, i.e. locally κ-presentable with $- \otimes -$ preserving κ-filtered colimits in each variable. As we saw, we define $FM = M \otimes M + I$, $GM = (M \otimes M) \otimes M + M + M$. Let's write $T(F)$, $T(G)$ for the free 2-monads on these 2-endofunctors.

There is a natural 2-functor $T(F)\text{-}\mathsf{Alg}_S \to T(G)\text{-}\mathsf{Alg}_S$ sending (M, p, u) to $(M, p \cdot (p \otimes u), p \cdot (u \otimes M))$ and there is another two functor mapping it to $(M, p \cdot (M \otimes p), \mathrm{id}_M, \mathrm{id}_M)$. These correspond to 2-monad morphisms and the 2-monad for monoids is exactly its coequalizer.

A relevant question: what would happen if we considered lax/pseudo T-morphisms in this case? The simple existence of $\{f, f\}_l$ tells us that this is some kind of equalizer, however there is a problem: what is $T(F)\text{-}\mathsf{Alg}_l$ and what does the 2-functor $T(F)\text{-}\mathsf{Alg}_l \to T(G)\text{-}\mathsf{Alg}_l$ look like?

From $T(F) \rightsquigarrow \{f, f\}_l$ we get a morphism $T \to T(F) \to \{f, f\}_l$, which is however hard to analyze. This requires a bit of a detour.

Theorem 0.2.2 (doctrinal adjunction)**.** Let $(f, \overline{f}): (A, a) \to (B, b)$ be a pseudo T-morphism s.t. f is a left adjoint to $u: B \to A$ with unit η and counit ϵ. Then there exists a unique lax T-morphism structure $\overline{u}$ on u s.t. η and ϵ are T-transformations.

Proof. We shall prove uniqueness. For this, we observe that the equality

$$
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[dd, equal] \arrow[ddr, phantom, "="] & A \arrow[dr, "f"] \arrow[dd, equal] \arrow[ddr, phantom, "\overset{\eta}{\Rightarrow}"] & \\
& & B \arrow[dl, "u"] \\
TA \arrow[r, "a"'] & A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TA \arrow[rr, "a"] \arrow[dr, "Tf"] \arrow[dd, equal] \arrow[d, phantom, "\overset{T\eta}{\Rightarrow}"] & & A \arrow[dr, "f"] \arrow[d, phantom, "\overset{\overline{f}}{\Rightarrow}"] & \\
{} & TB \arrow[rr, "b"] \arrow[dl, "Tu"] \arrow[dr, phantom, "\overset{\overline{b}}{\Rightarrow}"] & {} & B \arrow[dl, "u"] \\
TA \arrow[rr, "a"'] & & A &
\end{tikzcd}
$$

implies that (is the $\overline{f}$ inverted???)

$$
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[dr, equal] \arrow[dr, phantom, "\overset{T\eta}{\Rightarrow}"] & TB \arrow[r, "b"] \arrow[d, "Tu"] \arrow[dr, phantom, "\overset{\overline{u}}{\Rightarrow}"] & B \arrow[d, "u"] \\
& TA \arrow[r, "a"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
TA \arrow[r, "Tf"] \arrow[d, "a"] \arrow[dr, phantom, "\overset{\overline{f}^{-1}}{\Rightarrow}"] & TB \arrow[d, "b"] \\
A \arrow[r, "f"] \arrow[dr, equal] \arrow[dr, phantom, "\overset{\eta}{\Rightarrow}"] & B \arrow[d, "u"] \\
& A
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
TB \arrow[r, "b"] \arrow[d, "Tu"'] \arrow[dr, phantom, "\overset{\overline{u}}{\Rightarrow}"] & B \arrow[d, "u"] \\
TA \arrow[r, "a"'] & A
\end{tikzcd}
\quad = \quad \textit{draw}
$$

by the triangle identities for $Tf \dashv Tu$.

For existance, $(u, \overline{u})$ is a lax T-morphism with the desired properties by exercise 13.4 from the previous course. $\square$

We now study a kind of limit existing in T-Alg_l.

Definition 0.2.3. Given a 2-category $\mathcal{K}$ and an arrow $f : A \to B$ in it, it colax limit is the universal 2-cell

$$
\begin{tikzcd}
& C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
$$

in $\mathcal{K}$. This means that for each $a : X \to A$, $b : X \to B$ and $\alpha : f \cdot a \to b$ there exists a unique 1-cell $t : X \to C$ s.t.

$$
\begin{tikzcd}
& X \arrow[dl, "a"'] \arrow[dr, "b"] \arrow[d, phantom, "\alpha\Downarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& X \arrow[d, "t"] & \\
& C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
A \arrow[rr, "f"'] & {} & B
\end{tikzcd}
$$

holds. The 2-dimensional universal property asserts that for all $a' : A \to A$, $b' : X \to B$,

$\alpha' \colon b' \to f \cdot a'$ and 2-cells $\gamma \colon a \Rightarrow a'$, $\delta \colon b \Rightarrow b'$ with

$$
\begin{tikzcd}
X \arrow[d, bend right, "a'"'] \arrow[d, bend left, "a"] \arrow[d, phantom, "\overset{\gamma}{\Leftarrow}"] \arrow[drr, bend left, "b"] \arrow[drr, phantom, "\Downarrow\alpha"] & & \\
A \arrow[rr, bend right, "f"'] & & B
\end{tikzcd}
=
\begin{tikzcd}
& & X \arrow[dll, bend right, "a'"'] \arrow[d, bend right, "b'"'] \arrow[d, bend left, "b"] \arrow[d, phantom, "\overset{\delta}{\Leftarrow}"] \arrow[dll, phantom, "\overset{\alpha'}{\Longleftarrow}"] \\
A \arrow[rr, bend right, "f"'] & & B
\end{tikzcd}
$$

there exists a unique 2-cell $\phi \colon t \Rightarrow t'$ s.t. $p \cdot \phi = \gamma$, $q \cdot \phi = \delta$.

Notice that this is precisely the comma object

$$
\begin{tikzcd}
\mathrm{id} \downarrow f \arrow[r] \arrow[d] & B \arrow[d, equal] \arrow[dl, phantom, "\swarrow"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

in $\mathcal{K}$. This is a weighted limit in the enriched sense, hence defined via an isomorphism of categories and not just an equivalence.

The pseudo limit of f is the analogous construction with λ and α invertible. The lax limit has the direction of λ reversed.

We can now state the following.

Proposition 0.2.4. Let $\mathcal{K}$ be a 2-category with colax limits of arrows and T a 2-monad on it. For any 1-cell $(f, \overline{f}) \colon (A, a) \rightsquigarrow (B, b)$ in $T\text{-}\mathsf{Alg}_l$ there exists a unique T-algebra structure on the colax limit of f s.t. the projections are strict 2-morphisms. The 2-cell

$$
\begin{tikzcd}
& C \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
A \arrow[rr, squiggly, "f"'] & & B
\end{tikzcd}
$$

is a T-transformation and (G, λ) is a colax limit in $T\text{-}\mathsf{Alg}_l$. Moreover, p and q jointly detect strict morphisms, that is a 1-cell $t \colon X \to C$ is strict if and only if pt and qt are strict. In particular, the colax limit of $(f, \overline{f})$ exists and it is strictly presented by the forgetful 2-functor $U_l \colon T\text{-}\mathsf{Alg}_l \to \mathcal{K}$.

Proof. There exists a unique 1-cell $c \colon TC \to C$ s.t. the equation

$$
\begin{tikzcd}
& TA \arrow[r, "a"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overset{\overline{f}}{\Longrightarrow}"] & A \arrow[d, "f"] \\
TC \arrow[ur, "Tp"] \arrow[dr, "Tq"'] \arrow[r, phantom, "T\lambda \Uparrow"] & TB \arrow[r, "b"'] & B
\end{tikzcd}
=
\begin{tikzcd}
& & A \arrow[dd, "f"] \\
TC \arrow[r, "c"] & C \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\lambda\Uparrow"] & \\
& & B
\end{tikzcd}
$$

holds. Note that the direction of λ is important! Since $p \cdot c = a \cdot Tp$, $q \cdot c = b \cdot Tq$, so if we can show that (C, c) is a T-algebra then p and q are strict T-morphisms. Similarly, the above equation then says that λ is a T-transformation.

Applying T to the above equation and whiskering the result on the right with $\overline{f}$ gives

$$
\begin{tikzcd}
 & T^2A \arrow[r, "Ta"] \arrow[dd, "T^2f" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{T\overline{f}}"] & TA \arrow[r, "a"] \arrow[dd, "Tf" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{\overline{f}}"] & A \arrow[dd, "f"] \\
T^2C \arrow[ur, "T^2p"] \arrow[dr, "T^2q"'] \arrow[r, phantom, "\Uparrow T^2\lambda"] & & & \\
 & T^2B \arrow[r, "Tb"'] & TB \arrow[r, "b"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & TA \arrow[r, "a"] \arrow[dd, "Tf" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{\overline{f}}"] & A \arrow[dd, "f"] \\
T^2C \arrow[r, "Tc"] & TC \arrow[ur, "Tp"] \arrow[dr, "Tq"'] \arrow[r, phantom, "\Uparrow T\lambda"] & & \\
 & & TB \arrow[r, "b"'] & B
\end{tikzcd}
$$

Notice that the diagram on the right reduces to

$$
\begin{tikzcd}
 & & & A \arrow[dd, "f"] \\
T^2C \arrow[r, "Tc"] & TC \arrow[r, "c"] & C \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\Uparrow\lambda"] & \\
 & & & B
\end{tikzcd}
$$

and applying the axioms for a lax T-morphism and the 2-naturality of $\mu\colon T^2 \Rightarrow T$, we find that the left hand side above is

$$
\begin{tikzcd}
 & T^2A \arrow[r, "Ta"] \arrow[dd, "T^2f" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{T\overline{f}}"] & TA \arrow[r, "a"] \arrow[dd, "Tf" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{\overline{f}}"] & A \arrow[dd, "f"] \\
T^2C \arrow[ur, "T^2p"] \arrow[dr, "T^2q"'] \arrow[r, phantom, "\Uparrow T^2\lambda"] & & & \\
 & T^2B \arrow[r, "Tb"'] & TB \arrow[r, "b"] & B
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & & TA \arrow[r, "a"] \arrow[dd, "Tf" description] \arrow[r, phantom, shift right=6ex, "\xRightarrow{\overline{f}}"] & A \arrow[dd, "f"] \\
T^2C \arrow[r, "\mu_C"] & TC \arrow[ur, "Tp"] \arrow[dr, "Tq"'] \arrow[r, phantom, "\Uparrow T\lambda"] & & \\
 & & TB \arrow[r, "b"'] & B
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
 & & & A \arrow[dd, "f"] \\
T^2C \arrow[r, "\mu_C"] & TC \arrow[r, "c"] & C \arrow[ur, "p"] \arrow[dr, "q"'] \arrow[r, phantom, "\Uparrow\lambda"] & \\
 & & & B
\end{tikzcd}
$$

so from the 1-dimensional universal property it follows that $c \cdot \mu_C = c \cdot Tc$. The unit axiom is left as an exercise. To show that (C, c) is a T-algebra, p, q are strict morphisms and λ is a

T-transformation we have to check the universal properties. Consider a 2-cell

$$\begin{tikzcd} & X \arrow[dl, squiggly, "g"'] \arrow[dr, squiggly, "h"] \arrow[d, phantom, "\alpha\Downarrow"] & \\ A \arrow[rr, squiggly, "f"'] & & B \end{tikzcd}$$

in $T\text{-}\mathsf{Alg}_l$. This is a 2-cell $\alpha\colon h \Rightarrow fg$ in $\mathcal{K}$ subject to the axiom for a T-transformation. In particular, there exists a unique 1-cell $t\colon X \to C$ s.t. $\alpha = \lambda t$. The composite $\lambda\cdot c\cdot Tt$ corresponds to the 2-cell

$$\begin{tikzcd} & TA \arrow[r, "a"] \arrow[dd, "Tf"] \arrow[ddr, phantom, "\overset{\bar f}{\Rightarrow}"] & A \arrow[dd, "f"] \\ TX \arrow[ur, "Tg"] \arrow[dr, "Th"'] \arrow[r, phantom, "T\alpha\Uparrow"] & & \\ & TB \arrow[r, "b"'] & B \end{tikzcd}$$

and the composite $\lambda \cdot t \cdot x$ corresponds to the 2-cell

$$\begin{tikzcd} & & A \arrow[dd, "f"] \\ TX \arrow[r, "x"] & X \arrow[ur, "g"] \arrow[dr, "h"'] & \\ & & B \end{tikzcd}$$

in $\mathcal{K}$. Since α is a 2-cell in $T\text{-}\mathsf{Alg}_l$, comparing the first of these with $\bar g\colon a \cdot Tg \Rightarrow g \cdot x$, we get the 2-cell $\alpha \cdot x$ compared with $\bar h\colon b \cdot Th \Rightarrow h \cdot x$. In other words, $\bar g$ and $\bar h$ satisfy the defining equations for 2-cells in the 2-dimensional universal property of the colax limit of f. Thus there exists a unique 2-cell $\bar t\colon c \cdot Tt \Rightarrow t \cdot x$ s.t. $p \cdot \bar t = \bar g$ and $q \cdot \bar t = \bar h$. If we can show that $(t, \bar t)$ is a lax T-morphism, then these last equations show $p \cdot (t, \bar t) = (g, \bar g)$ and $q \cdot (t, \bar t) = (h, \bar h)$ as 1-cells in $T\text{-}\mathsf{Alg}_l$. Conversely, the equations also show that $(t, \bar t)$ is unique. As a diagram, the equation $p\bar t = \bar g$ looks like

$$\begin{tikzcd} & X \arrow[dr, "t"] \arrow[drr, bend left, "g"] \arrow[d, phantom, "\Uparrow\bar t"] & & \\ TX \arrow[ur, "x"] \arrow[dr, "Tt"'] \arrow[ddr, bend right, "Tg"'] & {} & C \arrow[r, "p"'] & A \\ & TC \arrow[ur, "c"'] \arrow[d, "Tp"] & & \\ & TA \arrow[uurr, bend right, "a"'] & & \end{tikzcd} = \begin{tikzcd} & X \arrow[dr, "g"] \arrow[dd, phantom, "\bar g\Uparrow"] & \\ TX \arrow[ur, "x"] \arrow[dr, "Tg"'] & & A \\ & TA \arrow[ur, "a"'] & \end{tikzcd}$$

in $\mathcal{K}$. Applying T to this equation and composing with Tg we get

$$
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "t"] \arrow[ddrr, bend left, "g"] \arrow[dd, phantom, "\Uparrow\bar{t}"] & & \\
 & TX \arrow[ur, "x"] \arrow[dr, "Tt"] \arrow[dd, phantom, "\Uparrow T\bar{t}"] & & C \arrow[dr, "p"] \arrow[dd, phantom, "\|"] \arrow[r, phantom, "/\!/"] & {} \\
T^2X \arrow[ur, "Tx"] \arrow[dr, "T^2t"] \arrow[ddrr, bend right, "T^2g"'] & & TC \arrow[ur, "c"] \arrow[dr, "Tp"] \arrow[dd, phantom, "\|"] & & A \\
 & T^2C \arrow[ur, "Tc"] \arrow[dr, "T^2p"] \arrow[d, phantom, "/\!/"] & & TA \arrow[ur, "a"] & \\
 & {} & T^2A \arrow[ur, "Ta"] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "g"] \arrow[d, phantom, "\bar{g}\Uparrow"] & \\
 & TX \arrow[ur, "x"] \arrow[dr, "Tg"] \arrow[d, phantom, "T\bar{g}\Uparrow"] & {} & A \\
T^2X \arrow[ur, "Tx"] \arrow[dr, "T^2g"'] & {} & TA \arrow[ur, "a"] & \\
 & T^2A \arrow[ur, "Ta"'] & &
\end{tikzcd}
.
$$

Using the fact that $(g, \bar{g})$ is a lax T-morphism and the 2-naturality of $\mu\colon T^2 \Rightarrow T$ we find that the above 2-cell is equal to

$$
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\bar{g}"] & \\
 & TX \arrow[ur, "x"] \arrow[d, "Tt"'] \arrow[dr, "Tg"] & {} & A \\
T^2X \arrow[ur, "\mu_X"] \arrow[dr, "T^2t"'] \arrow[r, phantom, "\|"] & TC \arrow[r, "Tp"'] \arrow[ur, phantom, "/\!/"] & TA \arrow[ur, "a"'] & \\
 & T^2C \arrow[u, "\mu_C"] \arrow[r, "T^2p"'] \arrow[ur, phantom, "\|"] & T^2A \arrow[u, "\mu_A"'] &
\end{tikzcd}
\underset{p\bar{t}=\bar{g}}{=^{\text{from}}}
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "t"] \arrow[d, phantom, "\Uparrow\bar{t}"] & & \\
 & TX \arrow[ur, "x"] \arrow[dr, "Tt"] & {} & C \arrow[r, "p"] & A \\
T^2X \arrow[ur, "\mu_X"] \arrow[dr, "T^2t"'] \arrow[rr, phantom, "\|"] & & TC \arrow[ur, "c"'] & & \\
 & T^2C \arrow[ur, "\mu_C"'] & & &
\end{tikzcd}
.
$$

A similar argument shows that the equality

$$
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "t"] \arrow[d, phantom, "\Uparrow\bar{t}"] & B \\
 & TX \arrow[ur, "x"] \arrow[dr, "Tt"] \arrow[d, phantom, "\Uparrow T\bar{t}"] & {} & C \arrow[u, "q"'] \\
T^2X \arrow[ur, "Tx"] \arrow[dr, "T^2t"'] & {} & TC \arrow[ur, "c"'] & \\
 & T^2C \arrow[ur, "Tc"'] & &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
 & & X \arrow[dr, "t"] \arrow[d, phantom, "\Uparrow\bar{t}"] & B \\
 & TX \arrow[ur, "x"] \arrow[dr, "Tt"] \arrow[d, phantom, "\|"] & {} & C \arrow[u, "q"'] \\
T^2X \arrow[ur, "\mu_X"] \arrow[dr, "T^2t"'] & {} & TC \arrow[ur, "c"'] & \\
 & T^2C \arrow[ur, "\mu_C"'] & &
\end{tikzcd}
$$

holds. From the uniqueness part of the 2-dimensional universal property it follows that the equation

$$
\begin{tikzcd}
T^2X \arrow[r, "Tx"] \arrow[d, "T^2t"'] \arrow[dr, phantom, "\overset{T\bar{z}}{\Longrightarrow}"] & TX \arrow[r, "x"] \arrow[d, "Tt"'] \arrow[dr, phantom, "\overset{\bar{z}}{\Longrightarrow}"] & X \arrow[d, "t"] \\
T^2C \arrow[r, "Tc"'] & TC \arrow[r, "c"'] & C
\end{tikzcd}
=
\begin{tikzcd}
T^2X \arrow[r, "\mu_X"] \arrow[d, "T^2t"'] & TX \arrow[r, "x"] \arrow[d, "Tt"'] & X \arrow[d, "t"] \\
T^2C \arrow[r, "\mu_C"'] & TC \arrow[r, "c"'] & C
\end{tikzcd}
$$

holds. The unit axiom is again left as an exercise. It remains to check the 2-dimensional universal property, so consider γ, δ 2-cells in $T\text{-}\mathsf{Alg}_l$ s.t.

$$
\begin{tikzcd}
X \arrow[dr, squiggly, "h"] \arrow[d, squiggly, bend right, "g'"'] \arrow[d, squiggly, bend left, "g"] \arrow[d, phantom, "\Lleftarrow\gamma"] & \\
A \arrow[r, squiggly, "f"'] \arrow[ur, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, squiggly, bend left, "h"] \arrow[r, squiggly, "h'"'] \arrow[r, phantom, bend left=20, "\Swarrow\delta"] \arrow[d, squiggly, "g'"'] & B \\
A \arrow[ur, squiggly, "f"'] \arrow[ur, phantom, "\alpha'\Swarrow"] &
\end{tikzcd}
$$

holds. The data of a T-transformation is just a 2-cell in $\mathcal{K}$ which is compared and whiskered as in $\mathcal{K}$. From the universal property of φ in $\mathcal{K}$ it follows that there is a unique 2-cell $\varphi\colon t \Rightarrow t'$ with $p\varphi = \gamma$, $q\varphi = \delta$. It only remains to check that φ is a T-transformation, i.e. that the equation

$$\begin{tikzcd} TX \arrow[r, "x"] \arrow[d, "Tt"'] \arrow[dr, phantom, "\overset{\bar{t}}{\Rightarrow}"] & X \arrow[d, bend right, "t"'] \arrow[d, bend left, "t'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] \\ TC \arrow[r, "c"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} TX \arrow[r, "x"] \arrow[d, bend right, "Tt"'] \arrow[d, bend left, "Tt'"] \arrow[d, phantom, "\overset{T\varphi}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\bar{t'}}{\Rightarrow}"] & X \arrow[d, "t'"] \\ TC \arrow[r, "c"'] & C \end{tikzcd}$$

holds. After whiskering with $p\colon C \to A$, the equation becomes

$$\begin{tikzcd} TX \arrow[r, "x"] \arrow[d, "Tg"'] \arrow[dr, phantom, "\overset{\bar{g}}{\Rightarrow}"] & X \arrow[d, bend right, "g"'] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} TX \arrow[r, "x"] \arrow[d, bend right, "Tg"'] \arrow[d, bend left, "Tg'"] \arrow[d, phantom, "\overset{T\gamma}{\Rightarrow}"] \arrow[dr, phantom, "\overset{\bar{g}'}{\Rightarrow}"] & X \arrow[d, "g'"] \\ TA \arrow[r, "a"'] & A \end{tikzcd}$$

which holds since γ is a T-transformation. The equation also holds after whiskering with q since δ is a T-transformation. Therefore φ is indeed a T-transformation, which concludes the proof of the 2-dimensional universal property. Finally, if q and h are strict T-morphisms, then the equation $p \cdot \bar{t} = \bar{g}$ and $q \cdot \bar{t} = \bar{h}$ implies that $\bar{t} = 1$, i.e. $(t, \bar{t})$ is a strict T-morphism. □

In any 2-category $\mathcal{K}$ with colax limits of arrows, we get for each $f\colon A \to B$ with colax limit (C_f, p_f, q_f, X) a unique 1-cell $r_f\colon A \to C_f$ s.t.

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, "f"] \arrow[d, phantom, "="] & \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd} \quad = \quad \begin{tikzcd} & A \arrow[d, "r_f"] & \\ & C_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda\Downarrow"] & \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

holds. In particular, $q_f r_f = f$ and $p_f r_f = \mathrm{id}_f$.

Proposition 0.2.5. In the above situation, there exists a unique 2-cell $\eta_f\colon \mathrm{id}_{C_f} \Rightarrow r_f \cdot p_f$ s.t. $p_f\eta_f = 1, q_f\eta_f = \lambda$. This 2-cell exhibits r_f as right adjoint of p_f with colimit the identity $p_f r_f = \mathrm{id}_A$.

Proof. Taking $\gamma = 1_{p_f}\colon p_f \Rightarrow p_f r_f p_f$ and $\delta = \lambda\colon q_f \Rightarrow fpf = q_f r_f p_f$ we have

$$\begin{tikzcd} A \arrow[d, "r_f"'] \arrow[dr, phantom, "="] & C_f \arrow[l, "p_f"'] \arrow[d, "p_f"'] \arrow[dr, "q_f"] \arrow[dr, phantom, "\overset{\lambda}{\Leftarrow}"] & \\ C_f \arrow[r, "p_f"'] & A \arrow[r, "f"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} & C_f \arrow[d, "p_f"'] \arrow[dr, bend left, "q_f"] \arrow[dr, phantom, "\lambda\Downarrow"] & \\ & A \arrow[dl, equal] \arrow[r, "f"] \arrow[d, phantom, "="] & B \\ A \arrow[urr, "f"'] & {} & \end{tikzcd}$$

$$= \qquad \begin{tikzcd} & C_f \arrow[d, "p_f"'] \arrow[dddr, bend left, "q_f"] & \\ & A \arrow[d, "r_f"'] \arrow[ddr, bend left, "f"] \arrow[r, phantom, "\Downarrow\lambda"] & {} \\ & C_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[r, phantom, "="] \arrow[d, phantom, "\lambda\Swarrow"] & {} \\ A \arrow[rr, "f"'] & {} & B \end{tikzcd}$$

so there exists a unique 2-cell $\eta_f\colon \mathrm{id}_{C_f} \Rightarrow r_f \cdot p_f$ with $p_f \cdot \eta_f = 1, q_f\eta_f = \lambda$ by the 2-dimensional universal property. It remains to show that the triangle identities hold. Since $\epsilon = 1$ these become $p_f\eta_f = 1$ and $\eta_f r_f = 1$. So one of these we already checked. For the second it suffices to check that it holds after whiskering with p_f and η_f, where we get $p_f\eta_f r_f = 1$ and $q_f\eta_f r_f = \lambda r_f = 1$ (by def of r_f) and $p_f\eta_f = 1$ by definition. □

A right adjoint r with counit the identity is sometimes called a RARI (Right Adjoint Right Inverse). The corresponding left adjoint is called a LALI (Left Adjoint Left Inverse). The dual concepts (with unit the identity) are called RALI and LARI. For $T\text{-}\mathsf{Alg}_p$ we can work instead with pseudolimits of arrows, which is the universal

$$\begin{tikzcd} & P_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\Downarrow\sim"] & \\ A \arrow[rr, "f"'] & {} & B. \end{tikzcd}$$

Proposition 0.2.6. The forgetful 2-functor $U_p\colon T\text{-}\mathsf{Alg}_p \to \mathcal{K}$ creates pseudolimits of arrows.

Proof. The same construction[1] as in the case of $T\text{-}\mathsf{Alg}_l$ works, we just have to observe that $\bar{t}$ is an isomorphism, which follows from $p\bar{t} = \bar{g}$ and $q\bar{t} = \bar{h}$ and the fact that those are isomorphisms, since f and g are pseudomorphisms and p, q jointly detect isos. □

Proposition 0.2.7. If $\mathcal{K}$ has pseudolimits of arrows and $(f, \bar{f})\colon A \rightsquigarrow B$ is a pseudo T-morphism, then there exists a unique $r_f\colon A \rightsquigarrow P_f$ such that

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[dr, squiggly, "f"] & \\ A \arrow[rr, squiggly, "f"'] & & B \end{tikzcd} \qquad = \qquad \begin{tikzcd} & A \arrow[d, squiggly, "r_f"] \arrow[ddl, equal, bend right] \arrow[ddr, squiggly, bend left, "f"] & \\ & P_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda\Downarrow\sim"] & \\ A \arrow[rr, squiggly, "f"'] & {} & B \end{tikzcd}$$

and an invertible $\eta_f\colon 1 \Rightarrow r_f p_f$ s.t. $(r_f, p_f, \eta_f, 1)$ is an adjoint equivalence.

Proof. Existence of η_f and triangle identities follow as before. Moreover, η_f is invertible since both $p_f\eta_f = 1$ and $q_f\eta_f = \lambda$ are invertible and p_f, q_f jointly detect isos. □

[1] In part p_f, q_f strict!

In particular, we can replace (up to equivalence) a pseudo T-morphism by a strict T-morphism

$$\begin{tikzcd} & P_f \arrow[dr, "q_f"] & \\ A \arrow[ur, rightsquigarrow, "r_f"] \arrow[rr, rightsquigarrow, "f"'] & \simeq & B \end{tikzcd}$$

of path-spaces. With this at hand we can prove the following theorem, which is useful for constructing 2-monads via presentations. Specifically, for identifying the pseudo and lax T-morphisms of such 2-monads.

Theorem 0.2.8. Let S and T be 2-monads on a 2-category with colax limits of arrows. Let $F_s\colon T\text{-}\mathsf{Alg}_s \to S\text{-}\mathsf{Alg}_s$ be a (strict) 2-functor such that the triangle

$$\begin{tikzcd} T\text{-}\mathsf{Alg}_s \arrow[rr, "F_s"] \arrow[dr, "U_t"'] & & S\text{-}\mathsf{Alg}_s \arrow[dl, "U_s"] \\ & \mathcal{K} & \end{tikzcd}$$

commutes. Then there exists a unique 2-functor $F_l\colon T\text{-}\mathsf{Alg}_l \to S\text{-}\mathsf{Alg}_l$ s.t. the diagram

$$\begin{tikzcd} T\text{-}\mathsf{Alg}_s \arrow[rr, "F_s"] \arrow[d, "J"'] & & S\text{-}\mathsf{Alg}_s \arrow[d, "J"] \\ T\text{-}\mathsf{Alg}_l \arrow[rr, "F_l"] \arrow[dr, "U_l"'] & & S\text{-}\mathsf{Alg}_l \arrow[dl, "U_l"] \\ & \mathcal{K} & \end{tikzcd}$$

commutes.

Proof. For the existence note that F_s is induced by a (unique) 2-monad morphism $\varphi\colon S \to T$ s.t. the semantics 1-functor

$$(-)\text{-}\mathsf{Alg}\colon 2\text{-}\,\mathsf{Mnd}(\mathcal{K})^{\mathrm{op}} \to 2\text{-}\mathbf{Cat}/\,\mathcal{K}$$

is full and faithful. This can be used to define F_l as follows. We send

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB \arrow[r, "b"'] \arrow[ur, phantom, "\overset{\bar f}{\Rightarrow}"] & B \end{tikzcd}$$

to

$$\begin{tikzcd} SA \arrow[r, "\varphi_A"] \arrow[d, "Sf"'] & TA \arrow[r, "a"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ SB \arrow[r, "\varphi_B"'] & TB \arrow[r, "b"'] \arrow[ur, phantom, "\overset{\bar f}{\Rightarrow}"] & B \end{tikzcd}$$

and we let F_l be the identity on 2-cells. The interesting part is the converse. Since the inclusions J are bijective on objects, F_l is uniquely determined on 0-cells. The two 2-functors $U_l\colon T\text{-}\mathsf{Alg} \to \mathcal{K}$ and $U_l\colon S\text{-}\mathsf{Alg}_l \to \mathcal{K}$ are both injective on 2-cells, so F_l is also uniquely determined on 2-cells.

It remains to show uniqueness on 1-cells. So let $(f, \bar{f})\colon (A, a) \rightsquigarrow (B, b)$ be a 1-cell in $T\text{-}\mathsf{Alg}_l$. Since we have colax limits of arrows in $\mathcal{K}$, we can factor $(f, \bar{f})$ as follows

$$
\begin{tikzcd}
 & A \arrow[dl, equal] \arrow[dr, squiggly, "f"] & \\
A \arrow[rr, squiggly, "f"'] & & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & A \arrow[ddl, equal, bend right] \arrow[d, squiggly, "r_f"] \arrow[ddr, squiggly, bend left, "f"] & \\
 & C_f \arrow[dl, "p_f"'] \arrow[dr, "q_f"] \arrow[d, phantom, "\lambda\Downarrow"] & \\
A \arrow[rr, squiggly, "f"'] & {} & B.
\end{tikzcd}
$$

It follows that $F_l(f, \bar{f}) = F_l(q_f) \circ F_l(r_f)$. Since the square in the diagram commutes, $F_l(q_f) = F_s(q_f)$, so it only remains to show that $F_l(r_f)$ is uniquely determined. We also know that $(r_f, p_f, \eta_f, 1)$ is an adjunction, so since F_l is a 2-functor it follows that $(F_l(r_f), F_l(p_f), F_l(\eta_f), 1)$ is an adjunction in $S\text{-}\mathsf{Alg}_s$. Since p_f is also strict, we have $F_l(p_f) = F_s(p_f)$. To summarize: $F_l(r_f)$ is a lax T-morphism structure on $U_l F_l(r_f) = U_l(r_f)$ so that η_f and 1 make it a right adjoint of $F_l(p_f)$ in $S\text{-}\mathsf{Alg}_l$. From the uniqueness part of doctrinal adjunction it follows that $F_l(r_f)$ is uniquely determined by $F_s(p_f), \eta_f, 1$. □

Remark 0.2.9. There is an analogous statement for $T\text{-}\mathsf{Alg}_p$ using the pseudolimit of arrows (assuming they exist in $\mathcal{K}$). Why is this useful? When dealing with monads given by presentations, we will (by construction) have a 2-functor $F_s\colon T(G)\text{-}\mathsf{Alg}_s \to T(F)\text{-}\mathsf{Alg}_s$, so a corresponding monad morphism $T(F) \to T(G)$, whenever $T(F), T(G)$ are free 2-monads on endofunctors F, G. So this corresponds to a 2-natural $F \to T(G)$, but it is in general hard to describe this explicitly. If we want to figure out what happens on lax morphisms from the definition, we would need to understand this instead. Usually it is easy to guess a 2-functor F_l that makes everything commute. This assumes that we have a description of $T(F)\text{-}\mathsf{Alg}_l$ purely in terms of F, which is indeed possible as we will see next.

Definition 0.2.10. Let $F\colon \mathcal{K} \to \mathcal{K}$ be a 2-functor. An F-algebra is a pair (A, a) with $a\colon FA \to A$ a 1-cell in $\mathcal{K}$ with no axioms. Strict morphisms $f\colon (A, a) \to (B, b)$ are 1-cells $f\colon A \to B$ s.t. $bFf = fa$. A lax F-morphism is a pair $(f, \bar{f})$ of a 1-cell $f\colon A \to B$ and a 2-cell

$$
\begin{tikzcd}
FA \arrow[r, "a"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & A \arrow[d, "f"] \\
FB \arrow[r, "b"'] & B
\end{tikzcd}
$$

subject to no axioms. An F-transformation $\rho\colon (f, \bar{f}) \Rightarrow (g, \bar{g})$ is a 2-cell $\rho\colon f \Rightarrow g$ s.t. the equation

$$
\begin{tikzcd}
FA \arrow[r, "a"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & A \arrow[d, bend right=50, "f"'] \arrow[d, bend left=50, "g"] \arrow[d, phantom, "\overset{\rho}{\Rightarrow}"] \\
FB \arrow[r, "b"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FA \arrow[r, "a"] \arrow[d, bend right=50, "Ff"'] \arrow[d, bend left=50, "Fg"] \arrow[d, phantom, "\overset{F\rho}{\Rightarrow}"] & A \arrow[d, "g"] \\
FB \arrow[r, "b"'] & B
\end{tikzcd}
$$

holds. We write $F\text{-}\mathsf{Alg}$ for the resulting 2-category. A pseudo F-morphism is an $(f, \bar{f})$ s.t. $\bar{f}$ is invertible and we write $F\text{-}\mathsf{Alg}_p$ for the corresponding 2-category.

As in the 1-dimensional case, we can relate F-algebras and $T(F)$-algebras.

Proposition 0.2.11. Let $\mathcal{K}$ be a locally presentable 2-category, F a κ-accessible 2-endofunctor on $\mathcal{K}$, $T(F)$ the free κ-accessible monad on F with universal 2-natural transformation $\psi\colon F \to T(F)$. We can then construct isomorphisms of categories

$$\psi^*\colon T(F)\text{-}\mathsf{Alg}_L \to F\text{-}\mathsf{Alg}_L$$

$$\psi^*\colon T(F)\text{-}\mathsf{Alg}_P \to F\text{-}\mathsf{Alg}_P$$

by whiskering with ψ.

Proof. It is clear that $FA \xrightarrow{\psi_a} T(F)A \xrightarrow{a} A$ is a F-algebra for any $T(F)$-algebra (A, a) and, for any lax $T(T)$-morphism $(f, \overline{f})$, the 2-cell

$$\begin{tikzcd}
FA \arrow[r, "\psi_a"] \arrow[d, "Ff"'] \arrow[dr, phantom, "//"] & T(F)A \arrow[r, "a"] \arrow[d, dashed, "T(F)f"] \arrow[dr, phantom, "\overset{\overline{f}}{\Longrightarrow}"] & A \arrow[d, "f"] \\
FB \arrow[r, "\psi_b"'] & T(F)B \arrow[r, "b"'] & B
\end{tikzcd}$$

is a lax F-morphism. Since composition of 1-cells in both $F\text{-}\mathsf{Alg}_L$ and $T(F)\text{-}\mathsf{Alg}_L$ is defined by attaching these 2-cells, this defines a functor on the underlying 1-categories.

Since ψ is 2-natural, the axiom for a $T(F)$-transformation turns into the axiom for a F-transformation, hence we can extend this to a 2-functor by acting as the identity on 2-cells.

It remains to show that this defines an isomorphism of 2-categories, or equivalently that it is a bijection on 0, 1 and 2-cells, which follows from the universal property of ψ.

Since ψ^* preserves the underlying 0, 1 and 2-cells we only need to check the bijection for a fixed underlying cell. In this case, the claim follows from the existence of the 2-monads $< A, A >$, $\{f, f\}_L$ and $[\rho, \rho]$. Namely, whiskering with ψ gives a bijection between 2-monad morphisms $T(F) \to < A, A >$ and mere 2-natural transformations $F \Rightarrow < A, A >$. By adjunction, this corresponds to $a\colon FA \to A$, subject to no axioms. The bijection on 1 and 2-cells follows analogously, as proof concerning $T(F)\text{-}\mathsf{Alg}_P$ and $F\text{-}\mathsf{Alg}_P$. $\square$

We can use this to identify $T\text{-}\mathsf{Alg}_L$ when T is given via a presentation through the following procedure. We start with various (accessible) 2-endofunctors $F, G \ldots$ on $\mathcal{K}$ and we construct 2-functors $F\text{-}\mathsf{Alg}_S \to G\text{-}\mathsf{Alg}_S$, etc. These are induced by monad morphisms $T(G) \to T(F)$ and if we want to know what happens on lax and pseudo morphisms we use 0.2.8.

Taking limits, we obtain new categories which are of the form $T\text{-}\mathsf{Alg}_S$ for the corresponding category of monads. We can then iterate this by considering 2-functors $T\text{-}\mathsf{Alg}_L \to W\text{-}\mathsf{Alg}_S$ for a 2-endofunctor W on $\mathcal{K}$.

To do this we need one more ingredient in order to identify the 2-category $(W \odot D)\text{-}\mathsf{Alg}_{S/L/P}$ for any small diagram $D\colon \mathcal{A}^{\mathrm{op}} \to 2\,\mathsf{Mnd}_\kappa(\mathcal{K})$ and any weight $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$.

For $T\text{-}\mathsf{Alg}_L$, this comes from the corresponding limit of 2-categories $\{W, D\text{-}\mathsf{Alg}_L\}$. To show it we first need to turn $(-)\text{-}\mathsf{Alg}_L$ into a 2-functor.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D}
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[r, "U^{\mathcal{D}}"] & \mathcal{K}
\end{tikzcd} \qquad = \qquad \mathrm{id}_{U^{\mathcal{C}}}$$

We now need to extend $(-)\text{-}\mathsf{Alg}_L$ to a 2-functor.

Recall that a monad modification $\alpha\colon \phi \Rrightarrow \psi$ between monad morphisms is a modification subject to two axioms.

The datum of a modification of 2-monads consists of a 2-cell α_A for each 0-cell $A \in \mathcal{K}$ and the axioms state that the equations

$$\begin{tikzcd} SSA \arrow[r, bend left, "S\phi_A"] \arrow[r, bend right, "S\psi_A"'] \arrow[r, phantom, "\Downarrow S\alpha_A"] & STA \arrow[r, bend left, "\phi_{TA}"] \arrow[r, bend right, "\psi_{TA}"'] \arrow[r, phantom, "\Downarrow \alpha_{TA}"] & TTA \arrow[r, "\mu^T_A"] & A \end{tikzcd} \quad = \quad \begin{tikzcd} SSA \arrow[r, "\mu^S_A"] & SA \arrow[r, bend left, "\phi_A"] \arrow[r, bend right, "\psi_A"'] \arrow[r, phantom, "\alpha_A\Downarrow"] & TA \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, "\eta^S_A"] & SA \arrow[r, bend left, "\phi_A"] \arrow[r, bend right, "\psi_A"'] \arrow[r, phantom, "\Downarrow\alpha_A"] & TA \end{tikzcd} \quad = \quad 1_{\eta^T_A}$$

hold, plus the modification axioms.

We want to finish extending $(-)\text{-}\mathsf{Alg}_L$ to a 2-functor $2-\mathsf{Mnd}_\kappa(\mathcal{K})^{\text{coop}} \to 2-\mathbf{CAT}/\mathcal{K}$, where the target has the 2-cells specified above, hence we have to define a 2-natural transformation $\alpha^*\colon \psi^* \Rightarrow \phi^*$ s.t. $U_L\alpha^* = 1$. Giving a 2-natural transformation means giving a 1-cell in $S\text{-}\mathsf{Alg}_L$ for each 0-cell in $T\text{-}\mathsf{Alg}_L$, i.e. for each T-algebra we have to specify a lax S-morphism.

We do it as follows: given $(A,a) \in T\text{-}\mathsf{Alg}_L$, we let $(\alpha^*)_{(A,a)}$ be the lax S-morphism

$$\begin{tikzcd} SA \arrow[r, "\psi_A"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha_A}{\Longrightarrow}"] & TA \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, "//"] & A \arrow[d, equal] \\ SA \arrow[r, "\phi_A"'] & TA \arrow[r, "a"'] & A \end{tikzcd}$$

with the identity as underlying 1-cell.

Proposition 0.2.12. The assignment $\alpha \mapsto \alpha^*$ is well-defined and thus $(-)\text{-}\mathsf{Alg}_L$ gives a 2-functor

$$2-\mathsf{Mnd}_\kappa(\mathcal{K})^{\text{coop}} \to 2-\mathbf{CAT}/\mathcal{K}$$

Proof. There are a few things to check. We leave some as exercises.

We start with one of the lax morphism axioms. We want to show that

$$(1) \qquad \begin{tikzcd} SSA \arrow[r, "S\psi_A"] \arrow[d, equal] \arrow[dr, phantom, "\overset{S\alpha_A}{\Longrightarrow}"] & STA \arrow[r, "Sa"] \arrow[d, equal] & SA \arrow[r, "\psi_A"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha_A}{\Longrightarrow}"] & TA \arrow[r, "a"] \arrow[d, equal] & A \arrow[d, equal] \\ SSA \arrow[r, "S\phi_A"'] & STA \arrow[r, "Sa"'] & SA \arrow[r, "\phi_A"'] & TA \arrow[r, "a"'] & A \end{tikzcd}$$

$$=$$

$$(2) \qquad \begin{tikzcd} SSA \arrow[r, "\mu^S_A"] \arrow[d, equal] & SA \arrow[r, "\psi_A"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha_A}{\Longrightarrow}"] & TA \arrow[r, "a"] \arrow[d, equal] & A \arrow[d, equal] \\ SSA \arrow[r, "\mu^S_A"'] & SA \arrow[r, "\psi_A"'] & TA \arrow[r, "a"'] & A \end{tikzcd}$$

Using a modification axiom,

$$
(1) \quad = \quad
\begin{tikzcd}
SSA \arrow[r, "S\psi_A"] \arrow[d, equal] \arrow[dr, phantom, "\xRightarrow{S\alpha_A}"] & STA \arrow[r, "Sa"] \arrow[d, equal] \arrow[dr, phantom, "\xRightarrow{\alpha_{TA}}"] & TTA \arrow[r, "Ta"] \arrow[d, equal] & TA \arrow[r, "a"] \arrow[d, equal] & A \arrow[d, equal] \\
SSA \arrow[r, "S\phi_A"'] & STA \arrow[r, "Sa"'] & TTA \arrow[r, "Ta"] \arrow[dr, "\mu_a"'] & TA \arrow[r, "a"] & A \\
 & & & TA \arrow[ur, "a"'] &
\end{tikzcd}
$$

and now we apply a monad modification axiom to find that this is equal to

$$
\begin{tikzcd}
SSA \arrow[r, "\mu^S_A"] & SA \arrow[r, bend left, "\phi_A"] \arrow[r, bend right, "\psi_A"'] \arrow[r, phantom, "\Downarrow\alpha_A"] & TA \arrow[r, "a"] & A
\end{tikzcd} ,
$$

which we can rewrite as (2). We leave the second axiom as an exercise.

Next we check the 2-naturality of α^*. For the 1-cell axiom, we need to consider a 1-cell $(f, \overline{f} \colon (A, a) \to (B, b)$ in $T\text{-}\mathsf{Alg}_L$. Then we have

$$
\begin{tikzcd}
\bullet \arrow[d, "\alpha^*_{(A,a)}"] \\
\bullet \arrow[d, "\phi^*_{(f,\overline{f})}"] \\
\bullet
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
SA \arrow[r, "\psi_A"] \arrow[d, equal] & TA \arrow[r, "a"] \arrow[d, equal] & A \arrow[d, equal] \\
SA \arrow[r, "\phi_A"] \arrow[d, "Sf"'] & TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\xRightarrow{\overline{f}}"] & A \arrow[d, "f"'] \\
SB \arrow[r, "\phi_B"'] & TB \arrow[r, "b"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
SA \arrow[r, "\psi_A"] \arrow[d, "Sf"'] & TA \arrow[r, "a"] \arrow[d, dashed, "Tf"] \arrow[dr, phantom, "\xRightarrow{\overline{f}}"] & A \arrow[d, "f"] \\
SB \arrow[r, bend left, "\psi_B"] \arrow[r, bend right, "\phi_B"'] \arrow[r, phantom, "\Uparrow\alpha_B"] & TB \arrow[r, "b"'] & B
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
SA \arrow[r, "\psi_A"] \arrow[d, "Sf"'] & TA \arrow[r, "a"] \arrow[d, "Tf"] & A \arrow[d, "f"] \\
SB \arrow[r, "\psi_B"'] \arrow[d, equal] & TB \arrow[r, "b"] \arrow[d, equal] & B \arrow[d, equal] \\
SB \arrow[r, "\phi_B"'] & TB \arrow[r, "b"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\bullet \arrow[d, "\psi^*_{(f,\overline{f})}"] \\
\bullet \arrow[d, "\alpha^*_{(B,b)}"] \\
\bullet
\end{tikzcd}
$$

which shows the 1-cell part of the 2-naturality condition. We leave the 2-cell part of 2-naturality as an exercise.

By construction, we have $U^L \alpha^*_{(A,a)} = 1_A$, so this really is a 2-cell in 2-$\mathbf{CAT}/\mathcal{K}$. This shows that this assignment extends to a 2-functor if we can prove that composition and whiskering operations for monad modifications turn into the corresponding operations in 2-$\mathbf{CAT}/\mathcal{K}$, which follows from the definition of composition and whiskering for modifications. $\square$

Remark 0.2.13. For $T\text{-}\mathsf{Alg}_p$ we only have 2-naturality for invertible modifications.

Next we want to check that $(-)\text{-}\mathsf{Alg}_l$ turns weighted colimits into weighted limits. For this we use the following characterization of $\langle A, A \rangle$, $\{f, f\}_l$ and $[\rho, \rho]$.

Proposition 0.2.14. Let $\mathcal{K}$ be complete and $A \in \mathcal{K}$. Then there is an isomorphism of categories

$$
\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}(T, \langle A, A \rangle) \to 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, T\text{-}\mathsf{Alg}_l \xrightarrow{U_l} \mathcal{K})
$$

which is 2-natural in T.

Proposition 0.2.15. The 2-category 2-$\mathbf{CAT}/\mathcal{K}$ is complete as a $\mathbf{Cat}$-enriched category.

Proof. For completeness we need conical limits and powers by $\mathbb{2} = \{0 \to 1\}$. We start with the latter. It is given by the pullback in 2-$\mathbf{CAT}$

$$\begin{tikzcd} \mathbb{2} \pitchfork U \arrow[r] \arrow[d, "V"'] \arrow[dr, phantom, "\lrcorner"] & \mathcal{C}^{\mathbb{2}} \arrow[d, "U^{\mathbb{2}}"] \\ \mathcal{K} \arrow[r, "\ulcorner \mathrm{id} \urcorner"'] & \mathcal{K}^{\mathbb{2}} \end{tikzcd}$$

where $\ulcorner \mathrm{id} \urcorner$ classifies the identity 2-cell on $\mathrm{id}_{\mathcal{K}}$. Note that there is a 2-dimensional aspect to this, which follows from the 2-dimensional universal property of $\mathcal{C}^{\mathbb{2}}$. We also have copowers by $\mathbb{2}$ given by $\mathcal{C} \times \mathbb{2} \xrightarrow{\mathrm{pr}} \mathcal{C} \xrightarrow{U} \mathcal{K}$, so we only need to check the 1-dimensional universal property for conical limits. Conical limits are classical: products are given by "wide" pullbacks

$$\begin{tikzcd} & \prod U_i \arrow[dl] \arrow[dr] & \\ \mathcal{C}_i \arrow[dr, "U_i"'] & \cdots & \mathcal{C}_j \arrow[dl, "U_j"] \\ & \mathcal{K} & \end{tikzcd}$$

while equalizer are computed as in 2-$\mathbf{CAT}$. $\square$

Now we have a 2-functor between complete 2-categories and we want to show that it preserves limits. The strategy is as follows.

Let $\mathcal{C}, \mathcal{D}$ be complete $\mathcal{V}$-categories, $F\colon \mathcal{C} \to \mathcal{D}$ a $\mathcal{V}$-functor, $D\colon \mathcal{A} \to \mathcal{C}$ a diagram and $\mathcal{W}\colon \mathcal{A} \to \mathcal{V}$ a weight. We get the comparison morphism $\bar{F}\colon F\{\mathcal{W}, \mathcal{D}\} \to \{\mathcal{W}, F\,\mathcal{D}\}$ in $\mathcal{D}$. We want to show that this is an iso. We will construct a new functor $G\colon \mathcal{D} \to \mathcal{E}$ s.t. both G and GF preserve weighted limits and G reflects isomorphisms. Then the comparison morphism $\bar{GF}\colon GF\{\mathcal{W}, \mathcal{D}\} \xrightarrow{\cong} \{\mathcal{W}, GF\}$ factors as $GF\{\mathcal{W}, \mathcal{D}\} \xrightarrow{G(\bar{F})} G\{\mathcal{W}, F\,\mathcal{D}\} \underset{\cong}{\xrightarrow{\bar{G}}} \{\mathcal{W}'\, GF\,\mathcal{D}\}$ so $G(\bar{F})$ is invertible hence also $\bar{F}$ is an isomorphism.

We want to construct such a functor G in our setting. For this we use the constructions $\langle A, A\rangle, \{f, f\}_l$ and $[\rho, \rho]$.

Proposition 0.2.16. Let $\mathcal{K}$ be complete. Then there is an isomorphism of categories, 2-natural in T.

$$2\text{-}\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}(T, \langle A, A\rangle) \to 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, T\text{-}\mathsf{Alg}_l \xrightarrow{U_l} \mathcal{K})$$

Proof. From Exercise 1.3 we know that there is a natural bijection between monad morphisms $T \to \langle A, A\rangle$ and T-$\mathsf{-Alg}$ structures $a\colon TA \to A$ on A. This gives the bijection on objects. Since this is constructed from the general theory of strict actions of strict monoidal categories, we know from Exercise 1.2 that monad modifications

$$\begin{tikzcd} T \arrow[r, bend left, "a_1"] \arrow[r, bend right, "a_2"'] \arrow[r, phantom, "\varphi\Downarrow"] & \langle A, A\rangle \end{tikzcd}$$

correspond to lax T-morphisms $(\mathrm{id}_A, \varphi)\colon (A, a_2) \to (A, a_1)$ (note the reversal of direction, omitted in the Exercise). This corresponds precisely to a 2-cell

$$\begin{tikzcd}
\mathbb{1} \arrow[rr, bend left, "{(A,a_2)}"] \arrow[rr, bend right, "{(A,a_1)}"'] \arrow[rr, phantom, "{(\mathrm{id},\varphi)\Downarrow}"] \arrow[dr, "A"'] & & T\text{-}\mathsf{Alg}_l \arrow[dl, "U_l"] \\
& \mathcal{K} &
\end{tikzcd}$$

in 2-$\mathbf{CAT}/\mathcal{K}$. $\square$

Proposition 0.2.17. Let $\mathcal{K}$ be a complete 2-category and $f\colon A \to B$ a 1-cell in $\mathcal{K}$. Then there is an isomorphism of categories

$$2\text{-}\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}(T, \{f, f\}_l) \to 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{2} \xrightarrow{f} \mathcal{K}, T\text{-}\mathsf{Alg}_l \xrightarrow{U_l} \mathcal{K})$$

which is 2-natural in T.

Proof. We already know this bijection on objects. From Exercise 1.4 we know that this bijection arises from the strict action $[\mathcal{K}, \mathcal{K}] \times \mathrm{Colax}[\mathbb{2}, \mathcal{K}] \to \mathrm{Colax}[\mathbb{2}, \mathcal{K}]$ of 2-categories. Using Exercise 1.2 here we find that monad modifications

$$\begin{tikzcd}
T \arrow[r, bend left, "\bar{f}_1"] \arrow[r, bend right, "\bar{f}_2"'] \arrow[r, phantom, "\xi\Downarrow"] & \{f, f\}_l
\end{tikzcd}$$

correspond to lax T-morphisms in $\mathrm{Colax}[\mathbb{2}, \mathcal{K}]$, which are the identity on objects, that is to pairs of 2-cells ξ_A, ξ_B s.t.

$$\begin{tikzcd}
TA \arrow[r, bend left, "a_2"] \arrow[r, bend right, "a_1"'] \arrow[r, phantom, "\Uparrow\xi_A"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\bar{f}_1}{\Rrightarrow}"] & A \arrow[d, "f"] \\
TB \arrow[r, "b_1"'] & B
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "a_2"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\bar{f}_2}{\Rrightarrow}"] & A \arrow[d, "f"] \\
TB \arrow[r, bend left, "b_2"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\Uparrow\xi_B"] & B
\end{tikzcd}$$

holds and $(\mathrm{id}_A, \xi_A)\colon (A, a_2) \to (A, a_1)$, $(\mathrm{id}_B, \xi_B)\colon (B, b_2) \to (B, b_1)$ are lax T-morphisms (exercise). This is precisely a 2-cell

$$\begin{tikzcd}
\mathbb{2} \arrow[rr, bend left, "{(f,\bar{f}_2)}"] \arrow[rr, bend right, "{(f,\bar{f}_1)}"'] \arrow[rr, phantom, "\Downarrow"] \arrow[dr, "f"'] & & T\text{-}\mathsf{Alg}_l \arrow[dl, "U_l"] \\
& \mathcal{K} &
\end{tikzcd}$$

in 2-$\mathbf{CAT}/\mathcal{K}$. $\square$

Proposition 0.2.18. If $\mathcal{K}$ is complete and $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\rho"] & B \end{tikzcd}$ 2-cell in $\mathcal{K}$, there is an iso of categories

$$2\text{-}\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}(T,[\rho,\rho]) \longrightarrow 2\text{-}\mathbf{CAT}/\mathcal{K}\left(\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd} \xrightarrow{\rho} \mathcal{K}, T\text{-}\mathsf{Alg}_l \xrightarrow{U_l} \mathcal{K}\right)$$

that is 2-natural in T.

Proof. One uses the action of $[\mathcal{K},\mathcal{K}]$ on

$$\mathrm{Colax}\left[\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}, \mathcal{K}\right],$$

which has objects the 2-cells $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\rho"] & B \end{tikzcd}$, morphisms the quadruples (a,ϕ,ψ,b) such that

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\rho}{\Rightarrow}"] & A' \arrow[d, "g'"] \\ B \arrow[r, "b"'] \arrow[ur, phantom, "\overset{\psi}{\Longrightarrow}"] & B' \end{tikzcd} = \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & A' \arrow[d, bend right, "f'"'] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\overset{\rho'}{\Rightarrow}"] \\ B \arrow[r, "b"'] \arrow[ur, phantom, "\overset{\varphi}{\Longrightarrow}"] & B' \end{tikzcd}$$

holds. The 2-cells are pairs of 2-cells subject to two axioms spelled out in the exercises. The construction is then analogous to the previous two propositions. That is we have to analyze what exactly a T-algebra in

$$\mathrm{Colax}\left[\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}, \mathcal{K}\right]$$

is and what a lax T-morphism is, whose 1-cell part is the identity. □

The existence of adjoints is due to the completeness assumption. With this at hand we can now prove that $(-)\text{-}\mathsf{Alg}_l$ turns colimits into limits.

Theorem 0.2.19. Let $\mathcal{K}$ be a locally κ-presentable 2-category. Then the 2-functor

$$(-)\text{-}\mathsf{Alg}_l \colon 2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})^{\mathrm{coop}} \to 2\text{-}\mathbf{CAT}/\mathcal{K}$$

turns weighted colimits into limits.

Proof. We already know that the inclusion $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K}) \to 2\text{-}\mathsf{Mnd}(\mathcal{K})$ preserves weighted colimits, so it suffices to prove the claim for diagrams in the latter 2-category, which happen to have a colimit. So let $D \colon \mathcal{A} \to 2\text{-}\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}$ be a diagram, $\mathcal{W}\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{CAT}$ a weight such that $\mathcal{W} \odot_{\mathcal{A}} D$ exists in $2\text{-}\mathsf{Mnd}(\mathcal{K})^{\mathrm{co}}$. We have a comparison morphism $L\colon \mathcal{W} \odot_{\mathcal{A}} D\text{-}\mathsf{Alg}_l \to \{\mathcal{W}, D\text{-}\mathsf{Alg}_l\}$ in $2\text{-}\mathbf{CAT}/\mathcal{K}$. The represented 2-functor $2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, -)$ preserves weighted limits (as homs do) and the composite $2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, -) \circ (-)\text{-}\mathsf{Alg}_l$ also preserves weighted limits, since it is represented by $\langle A, A\rangle$ by the first Proposition above. So in the commuting diagram

$$\begin{tikzcd} 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, \mathcal{W} \odot_{\mathcal{A}} D\text{-}\mathsf{Alg}_l) \arrow[rr, "2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1}\xrightarrow{A}\mathcal{K},L)"] \arrow[dr, "\cong", "\text{comparison}"'] & & 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, \{\mathcal{W}, D\text{-}\mathsf{Alg}_l\}) \arrow[dl, "\cong"', "\text{comparison}"] \\ & \{\mathcal{W}, 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, D\text{-}\mathsf{Alg}_l)\} & \end{tikzcd}$$

both arrows labelled "comparison" are isomorphisms (compare with the discussion above for $F = (-)\text{-}\mathsf{Alg}_l, G = 2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, -)$). Upshot: for each $A \in \mathcal{K}$, $2\text{-}\mathbf{CAT}/\mathcal{K}(\mathbb{1} \xrightarrow{A} \mathcal{K}, L)$ is an isomorphism. Using the same argument applied to $(0 \to 1) \xrightarrow{f} \mathcal{K}$ and

$$\begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \arrow[r, "\rho"] & \mathcal{K} \end{tikzcd}$$

and the propositions about $\{f, f\}_l$ and $[\rho, \rho]$ we find that for all 1-cells f and all 2-cells ρ the 2-functors $2\text{-}\mathbf{CAT}/\mathcal{K}(f, L)$ and $2\text{-}\mathbf{CAT}/\mathcal{K}(\rho, L)$ are isomorphisms. Since the 2-functors $2\text{-}\mathbf{CAT}/\mathcal{K}(A, -), 2\text{-}\mathbf{CAT}/\mathcal{K}(f, -)$ and $2\text{-}\mathbf{CAT}/\mathcal{K}(\rho, -)$ jointly detect isomorphisms, we find that L is an isomorphism. □

We can do the same construction for $(-)\text{-}\mathsf{Alg}_p$ and $(-)\text{-}\mathsf{Alg}_s$. On the other hand, once we know that a 2-category is of the form $T\text{-}\mathsf{Alg}_l$ it has subcategories $T\text{-}\mathsf{Alg}_p$ and $T\text{-}\mathsf{Alg}_s$. We would like to be able to identify these in terms of the categories $D_i\text{-}\mathsf{Alg}_p, D_i\text{-}\mathsf{Alg}_s$ when forming limits. To do this we will use the 2-monads $\{f, f\}_p$ and $\{f, f\}_s$. We have 2-monads morphisms $\{f, f\}_s \to \{f, f\}_p \to \{f, f\}_l$ defined by the requirement that the 2-cell $\bar{f}$ is either an identity or an isomorphism. A factorization of $T \to \{f, f\}_l$ through one of these is unique, if it exists, which it does if and only if the lax morphism corresponding to φ is strict resp. pseudo.

Lemma 0.2.20. Given a diagram $D\colon \mathcal{A} \to 2 - \mathsf{Mnd}_\kappa(\mathcal{K})$ and a weight $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$, let $\mathcal{K}_i\colon D_i \to W \odot_{\mathcal{A}} D$ jointly "codetect" identities and isomorphisms. A 2-cell $\begin{tikzcd} W \odot_{\mathcal{A}} D \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & T \end{tikzcd}$ is an identity (an isomorphism) if and only if each $\alpha\mathcal{K}_i$ is. Then a lax $W \odot_{\mathcal{A}} D$-morphism $(f, \overline{f})$ is pseudo (strict) if and only if $(\mathcal{K}_i)^*(f, \overline{f})$ is.

Proof. This follows from the existence of the classifiers $\{f, f\}_S$, $\{f, f\}_P$, which are defined by the universal requirement that a certain 2-cell is an identity (an isomorphism). □

Lemma 0.2.21. The morphisms $\coprod_{i\in\mathcal{A}} \coprod_{w\in W_i} D_i \to W \odot_{\mathcal{A}} D$ jointly codetect isomorphisms and identities.

Proof. Applying $2 - \mathsf{Mnd}_\kappa(-T)$, this translates to a statement about weighted limits in $\mathbf{Cat}$, namely that for any $D'\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ the functor

$$\{W, D'\} \to \Pi_{i\in\mathcal{A}}\{W_i, D'_i\} \to \Pi_{i\in\mathcal{A}}\Pi_{w\in W_i} D_i$$

detects isomorphisms and identities, where the first is the canonical map we get from the characterization of weighted limits in terms of powers, products and equalizers and the second is a product of functors $\{W_i, D'_i\} = \mathrm{Fun}(W_i, D'_i) \to \mathrm{Fun}(\mathrm{Ob}\, W_i, D'_i)$. The latter functors detect isomorphisms and identities because a natural transformation is an isomorphism (an identity) if and only if all of its components are.

The first functor is the equalizer in the standard presentation of $\{W, D'\}$ in $\mathbf{Cat}$, hence a (not necessarily full) inclusion of subcategories, thus it detects identities (using injectivity on objects). It also detects isomorphisms: if $Ff = Gf$ and f is an isomorphism then $(Ff)^{-1} = (Gf)^{-1}$, so the inverse of an isomorphism lies in the equalizer. □

Remark 0.2.22. In practice once can do much better than the morphism in the above lemma: for example, for the cocomma object

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd},$$

the canonical arrow $B+C \to D$ codetects identities and isomorphisms. Identifying such a subset of objects with this property is easy once the 2-dimensional universal property is understood.

Summarizing, a lax $W \odot_{\mathcal{A}} D$-morphism consists of certain 2-cells involving the categories $D_i\text{-}\mathsf{Alg}_L$ and it will be pseudo (strict) if and only if all of the constituents are.

We now have almost all the ingredients necessary to identify $T\text{-}\mathsf{Alg}_{S/P/L}$ when T is given by a presentation.

Example 0.2.23. Consider a locally κ-presentable monoidal 2-category $\mathcal{K}$ such that for all objects x both $x \otimes -$ and $- \otimes x$ preserve κ-filtered colimits. Monoidal here means exactly the 1-categorical definition, replacing functors and natural transformations with their 2-dimensional counterparts. Examples of this are $[\mathcal{K}, \mathcal{K}]_\kappa$ with $\otimes = \circ$, $\mathcal{V} - \mathbf{Cat}$ for a lfp cosmos $\mathcal{V}$.

We now present a complete characterization of the 2-category of monoids on $\mathcal{K}$.

Let $F\colon \mathcal{K} \to \mathcal{K}$ be the 2-endofunctor $M \mapsto M \otimes M + I$. Then $T\text{-}\mathsf{Alg}_L$ has as objects the triples $(M, p\colon M \otimes M \to M, u\colon I \to M)$ subject to no axioms; morphisms $(M, p, u) \to (M', p', u')$ are 1-cells $f\colon M \to M'$ with 2-cells

$$
\begin{tikzcd}
M \otimes M + I \arrow[r, "p+u"] \arrow[d, "f\otimes f+I"'] \arrow[dr, phantom, "\xRightarrow{\overline{f}}"] & M \arrow[d, "f"] \\
M' \otimes M' + I \arrow[r, "p'+u'"'] & M'
\end{tikzcd}
$$

subject to no axioms. This amounts to a pair of 2-cells $\overline{f_2}\colon p' \cdot f \otimes f \Rightarrow f \cdot p$ and $\overline{f_0}\colon u' \to f \cdot u$ by the universal property of the coproduct. The 2-cells $(f, \overline{f_2}, \overline{f_0}) \Rightarrow (g, \overline{g_2}, \overline{g_0})$ are 2-cells $\phi\colon f \Rightarrow g$ s.t.

$$
\begin{tikzcd}
M \otimes M \arrow[r, "p"] \arrow[d, bend right, "f\otimes f"'] \arrow[d, bend left, "g\otimes g"] \arrow[d, phantom, "\overset{\phi\otimes\phi}{\Rightarrow}"] \arrow[dr, phantom, "\xRightarrow{\overline{g_2}}"] & M \arrow[d, "g"] \\
M' \otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}
=
\begin{tikzcd}
M \otimes M \arrow[r, "p"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_2}}"] & M \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] \\
M' \otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& M \arrow[dd, bend right, "f"'] \arrow[dd, bend left, "g"] \arrow[dd, phantom, "\overset{\phi}{\Rightarrow}"] \\
I \arrow[ur, "u"] \arrow[dr, "u'"'] \arrow[r, phantom, "\xRightarrow{\overline{f_0}}"] & \\
& M'
\end{tikzcd}
=
\begin{tikzcd}
& M \arrow[dd, "g"] \\
I \arrow[ur, "u"] \arrow[dr, "u'"'] \arrow[r, phantom, "\xRightarrow{\overline{g_0}}"] & \\
& M'
\end{tikzcd}
$$

hold.

Let $G\colon \mathcal{K} \to \mathcal{K}$ be the 2-endofunctor $GM = M \otimes (M \otimes M) + M + M$. The 2-category $G\text{-}\mathsf{Alg}_L$ has objects $(M, p_\alpha\colon M \otimes (M \otimes M) \to M, p_\lambda\colon M \to M, p_\rho\colon M \to M)$ and 1-cells are the quadruples $(f, \overline{f_\alpha}\colon p'_\alpha \cdot (f \otimes (f \otimes f)) \Rightarrow f \cdot p_\alpha, \overline{f_\lambda}\colon p'_\lambda \cdot f \Rightarrow f \cdot p_\lambda, \overline{f_\rho}\colon p'_\rho \cdot f \Rightarrow f \cdot p_\rho)$. The 2-cells are 2-cells $\phi\colon f \Rightarrow g$ s.t.

$$
\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "p_\alpha"] \arrow[d, bend right, "f\otimes(f\otimes f)"'] \arrow[d, bend left, "g\otimes(g\otimes g)"] \arrow[d, phantom, "\overset{\phi\otimes(\phi\otimes\phi)}{\Longrightarrow}"] \arrow[dr, phantom, "\xRightarrow{\overline{g_\alpha}}"] & M \arrow[d, "g"] \\
M' \otimes (M' \otimes M') \arrow[r, "p'_\alpha"'] & M'
\end{tikzcd}
=
\begin{tikzcd}
M \otimes (M \otimes M) \arrow[r, "p_\alpha"] \arrow[d, "f\otimes(f\otimes f)"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_\alpha}}"] & M \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \arrow[d, phantom, "\overset{\phi}{\Rightarrow}"] \\
M' \otimes (M' \otimes M') \arrow[r, "p'_\alpha"'] & M'
\end{tikzcd}
$$

and the other axioms hold. The pseudo/strict versions of these are the ones where $\overline{f_0}$, $\overline{f_2}$ (respectively $\overline{f_\alpha}$, $\overline{f_\lambda}$, $\overline{f_\rho}$) are isomorphisms/identities.

Next we construct two 2-functors $\psi_i\colon F\text{-}\mathsf{Alg}_L \to G\text{-}\mathsf{Alg}_L$, which send (M,p,u) to $(M, p\cdot M\otimes p, p\cdot u\otimes\cdot\lambda_M^{-1}, p\cdot M\otimes u\cdot\rho_M^{-1})$ and $(M, p\cdot p\otimes M\cdot\alpha_{M,M,M}, \mathrm{id}_M, \mathrm{id}_M)$ respectively.

On 1-cells, ψ_1 sends $(f,\overline{f_2},\overline{f_0})$ to

$$\begin{tikzcd}
M\otimes(M\otimes M) \arrow[r, "M\otimes p"] \arrow[d, "f\otimes(f\otimes f)"'] \arrow[dr, phantom, "\overset{f\otimes\overline{f_2}}{\Longrightarrow}"] & M\otimes M \arrow[r, "p"] \arrow[d, dashed, "f\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & M \arrow[d, "f"] \\
M'\otimes(M'\otimes M') \arrow[r, "M'\otimes p'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd},
\qquad
\begin{tikzcd}
M \arrow[r, "\lambda_M^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & I\otimes M \arrow[r, "u\otimes M"] \arrow[d, dashed, "I\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_0}\otimes f}{\Longrightarrow}"] & M\otimes M \arrow[r, "p"] \arrow[d, dashed, "f\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & M \arrow[d, "f"] \\
M' \arrow[r, "\lambda_{M'}^{-1}"'] & I\otimes M' \arrow[r, "u'\otimes M'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[r, "\rho_M^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & M\otimes I \arrow[r, "M\otimes u"] \arrow[d, dashed, "I\otimes f"] \arrow[dr, phantom, "\overset{f\otimes\overline{f_0}}{\Longrightarrow}"] & M\otimes M \arrow[r, "p"] \arrow[d, dashed, "f\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & M \arrow[d, "f"] \\
M' \arrow[r, "\rho_{M'}^{-1}"'] & M'\otimes I \arrow[r, "M'\otimes u'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}$$

The 2-functor ψ_2 sends $(f,\overline{f_2},\overline{f_0})$ to

$$\begin{tikzcd}
M\otimes(M\otimes M) \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "f\otimes(f\otimes f)"'] \arrow[dr, phantom, "="] & (M\otimes M)\otimes M \arrow[r, "p\otimes M"] \arrow[d, dashed, "(f\otimes f)\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_2}\otimes f}{\Longrightarrow}"] & M\otimes M \arrow[r, "p"] \arrow[d, dashed, "f\otimes f"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & M \arrow[d, "f"] \\
M'\otimes(M'\otimes M') \arrow[r, "\alpha_{M',M',M'}"'] & (M'\otimes M')\otimes M' \arrow[r, "p'\otimes M'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd},$$

1_f and 1_f.

On 2-cells both ψ_1 and ψ_2 act as the identity. The axioms hold because the α, λ, ρ parts are built from $\overline{f_0}$ and $\overline{f_2}$.

From the construction we see that the ψ_i restrict to 2-functors $F\text{-}\mathsf{Alg}_S \to G\text{-}\mathsf{Alg}_S$ and these restrictions are induced by 2-monad morphisms $\hat{\psi_i}\colon T(G)\to T(F)$, the free 2-monads on G and F respectively, by full faithfullness of the 1-functor $(-)\text{-}\mathsf{Alg}_S$. In other words, $\psi_i = (\hat{\psi_i})^*$ is a strict morphism. Since there is a unique extension of $(\hat{\psi_i})^*$ to a 2-functor on $T(F)\text{-}\mathsf{Alg}_L$ compatible with U_L, we have $\psi_i = (\hat{\psi_i})^*$ on all of $F\text{-}\mathsf{Alg}_L \cong T(F)\text{-}\mathsf{Alg}_L$.

Now let Mon be the coequalizer of $\hat{\psi_1}$ and $\hat{\psi_2}$ in $2-\mathsf{Mnd}_\kappa(\mathcal{K})$. Then $\mathsf{Mon}\text{-}\mathsf{Alg}_L$ is the coequalizer of the ψ_i, so the objects are precisely the monoids in $\mathcal{K}$, the 1-cells are the triples $(f,\overline{f_2},\overline{f_0})$ subject to three axioms, namely that the 2-cells depicted above are equal. The 2-cell axioms remain the same: compatibility with $\overline{f_2}$ and $\overline{f_0}$. The pseudo/strict morphisms are the ones where $\overline{f_2}$, $\overline{f_0}$ are invertible/identities, since $T(F)\to\mathsf{Mon}$ codetects isomorphisms/identities.

We can spell out what this means for κ-accessible monads.

Lax morphisms $(T,\mu^T,\eta^T)\to(S,\mu^S,\eta^S)$ are triples $(f,\overline{f_2},\overline{f_0})$ where $f\colon T\to S$ is 2-natural and $\overline{f_0}$, $\overline{f_2}$ are modifications

$$\begin{tikzcd}
& T \arrow[dd, "f"] & T^2 \arrow[l, "\mu^T"'] \arrow[dd, "f^2"] \\
\mathrm{id}_{\mathcal{K}} \arrow[ur, "\eta^T"] \arrow[dr, "\eta^S"'] \arrow[r, phantom, "\overset{\overline{f_0}}{\Longrightarrow}"] & {} \arrow[r, phantom, "\overset{\overline{f_2}}{\Longleftarrow}"] & {} \\
& S & S^2 \arrow[l, "\mu^S"]
\end{tikzcd}$$

such that –diagrams– hold.

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu^T"] \arrow[d, "Tf^2"'] \arrow[dr, phantom, "\overset{T\overline{f_2}}{\Longrightarrow}"] & T^2 \arrow[r, "\mu^T"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & T \arrow[dd, "f"] \\
TS^2 \arrow[r, "T\mu^S"] \arrow[d, "fS^2"'] \arrow[dr, phantom, "="] & TS \arrow[d, "fS"] & \\
S^3 \arrow[r, "S\mu^S"'] & S^2 \arrow[r, "\mu^S"'] & S
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
T^3 \arrow[r, "\mu^T T"] \arrow[d, "T^2 f"'] \arrow[dr, phantom, "="] & T^2 \arrow[r, "\mu^T"] \arrow[d, "Tf"] \arrow[dr, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & T \arrow[dd, "f"] \\
T^2 S \arrow[r, "\mu^T S"] \arrow[d, "f^2 S"'] \arrow[dr, phantom, "\overset{\overline{f_2}S}{\Longrightarrow}"] & TS \arrow[d, "fS"] & \\
S^3 \arrow[r, "\mu^S S"'] & S^2 \arrow[r, "\mu^S"'] & S
\end{tikzcd},
$$

$$
\begin{tikzcd}
 & T^2 \arrow[r, "\mu^T"] \arrow[d, "fT"] & T \arrow[dd, "f"] \\
T \arrow[ur, "\eta^T T"] \arrow[r, "\eta^S T"'] \arrow[ur, phantom, "\overline{f_0}"'] \arrow[d, "f"'] \arrow[dr, phantom, "="] & ST \arrow[d, "Sf"] \arrow[r, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & \\
S \arrow[r, "\eta^S S"'] & S^2 \arrow[r, "\mu^S"'] & S
\end{tikzcd}
\quad = \quad 1_f
$$

and

$$
\begin{tikzcd}
T \arrow[r, "T\eta^T"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & T^2 \arrow[r, "\mu^T"] \arrow[d, "fT"] & T \arrow[dd, "f"] \\
S \arrow[r, "S\eta^T"] \arrow[dr, "S\overline{f_0}"'] \arrow[r, phantom, "S\overline{f_0}"'] & ST \arrow[d, "Sf"] \arrow[r, phantom, "\overset{\overline{f_2}}{\Longrightarrow}"] & \\
 & S^2 \arrow[r, "\mu^S"'] & S
\end{tikzcd}
\quad = \quad 1_f
$$

hold.

Monad modifications between these are required to be compatible with $\overline{f_0}$ and $\overline{f_2}$.

It is somewhat surprising that these are really the lax morphisms if you try to recognize them without the machinery we built.

Next we want to describe the 2-monad for pseudomonoids in $\mathcal{K}$, which are "monoids up to coherent isomorphism", like monoidal $\mathcal{V}$-categories. Instead of forming the equalizer above, we form the iso-inserter and then we use an equifier to impose the coherence laws. An equifier universally makes two 2-cells equal.

Since this diagram will involve 2-cells, we need to know that all these 2-cells in $2-\mathbf{CAT}/\mathcal{K}$ come from 2-monad modifications. More precisely, we use the following.

Proposition 0.2.24. Let $\mathcal{K}$ be a locally κ-presentable 2-category. Then the 2-functor

$$(-)\text{-}\mathsf{Alg}_L\colon 2-\mathsf{Mnd}_\kappa(\mathcal{K})^{\mathrm{coop}} \to 2-\mathbf{CAT}/\mathcal{K}$$

is locally fully faithful: any 2-cell $\alpha\colon \phi^* \Rightarrow \psi^*$ comes from a unique monad modification $\psi \Rightarrow \phi\colon S \to T$.

Proof. We reduce this to the fact that the semantics-structure adjunction is fully faithful in the 1-categorical case.

By the universal property of powers, α corresponds to a unique 2-functor

$$T\text{-}\mathsf{Alg}_L \xrightarrow{\ulcorner\alpha\urcorner} [2] \pitchfork S\text{-}\mathsf{Alg}_L \cong (S \odot [2])\text{-}\mathsf{Alg}_L$$

and $\ulcorner\alpha\urcorner$ sends strict T-morphisms to strict $(S \odot [2])$-morphisms. The inclusion $\{0,1\} \to [2]$ induces $[2] \pitchfork S\text{-}\mathsf{Alg}_L \xrightarrow{(\pi_1,\pi_2)} S\text{-}\mathsf{Alg}_L \times S\text{-}\mathsf{Alg}_L$ and we have $\pi_1\ulcorner\alpha\urcorner = \phi^*$, $\pi_2\ulcorner\alpha\urcorner = \psi^*$, thus the composite $(\pi_1,\pi_2)\ulcorner\alpha\urcorner$ sends strict T-morphisms to strict $S+S$-morphisms.

Since this inclusion codetects identities it follows that (π_1,π_2) detects strict morphisms, so $\ulcorner\alpha\urcorner$ does indeed send strict T-morphisms to strict $S \odot [2]$-morphisms. The restriction to strict morphisms comes from a 2-monad morphism γ. Moreover, by the uniqueness of the extension to lax morphisms we must have $\ulcorner\alpha\urcorner = \gamma^*$ on all of $T\text{-}\mathsf{Alg}_L$. Thus, $\gamma\colon S \odot [2] \to T$ gives the desired 2-cell $\beta\colon \psi \Rightarrow \phi\colon S \to T$ woth $\beta^* = \alpha$ by construction.

This shows that $(-)\text{-}\mathsf{Alg}_L$ is full on 2-cells. Faithfullness again follows from the existence of $S \odot [2]$ and faithfulness of $(-)\text{-}\mathsf{Alg}_L$ on 1-cells: if β, β' induce the same 2-cell, then the corresponding $\ulcorner\beta\urcorner, \ulcorner\beta'\urcorner\colon S \odot [2] \to T$ induce the same 1-cell on $T\text{-}\mathsf{Alg}_L \to S \odot [2]\text{-}\mathsf{Alg}_L$, so they are in particular equal on $T\text{-}\mathsf{Alg}_S$, hence $\ulcorner\beta\urcorner = \ulcorner\beta'\urcorner$, so $\beta = \beta'$ by universal property of $S \odot [2]$. □

Remark 0.2.25. This argument would be simpler if $(-)\text{-}\mathsf{Alg}_L$ were fully faithful on 1-cells, but we don't know if this is true.

With this proposition in hand, we can now complete the construction of the 2-monad for pseudomonoids. Namely, instead of forming the coequalizer of $\hat{\psi}_1$ and $\hat{\psi}_2$ above, we form the co-iso-inserter T_1 in $2 - \mathsf{Mnd}_\kappa(\mathcal{K})$ instead.

Then $T_1\text{-}\mathsf{Alg}_L$ has objects (M,p,u,l), where l is an identity-on-objects isomorphism between $\psi_1(M,p,u)$ and $\psi(M,p,u)$. This amounts to giving invertible 2-cells

$$\begin{tikzcd}
M \otimes (M \otimes M) \arrow[rr, "M\otimes p"] \arrow[d, "\alpha_{M,M,M}"'] \arrow[drr, phantom, "\Rightarrow \alpha^M"] & & M \otimes M \arrow[d, "p"] \\
(M \otimes M) \otimes M \arrow[r, "p\otimes M"'] & M \otimes M \arrow[r, "p"'] & M
\end{tikzcd},$$

$$\begin{tikzcd}
M \arrow[r, "\lambda_M^{-1}"] \arrow[d, equal] \arrow[drrr, phantom, "\Rightarrow \lambda^M"] & I \otimes M \arrow[r, "u\otimes M"] & M \otimes M \arrow[r, "p"] & M \arrow[d, equal] \\
M \arrow[rrr, "\mathrm{id}_M"'] & & & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[r, "\rho_M^{-1}"] \arrow[d, equal] \arrow[drrr, phantom, "\Rightarrow \rho^M"] & M \otimes I \arrow[r, "M\otimes u"] & M \otimes M \arrow[r, "p"] & M \arrow[d, equal] \\
M \arrow[rrr, "\mathrm{id}_M"'] & & & M
\end{tikzcd}$$

subject to no axioms since l is a 2-cell in $G\text{-}\mathsf{Alg}_P$.

A 1-cell in $T_1\text{-}\mathsf{Alg}_L$ is a 1-cell $(f, \overline{f_0}, \overline{f_2})$ in $F\text{-}\mathsf{Alg}_L$ and that the resulting "naturality square" in G-Alg_L coming from l and l' commute (see the exercises). This means that the equations

$$
\begin{tikzcd}
M\otimes(M\otimes M) \arrow[rr, "M\otimes p"] \arrow[d, equal] & \arrow[d, phantom, "\xRightarrow{\alpha_{M,M,M}}"] & M\otimes M \arrow[r, "p"] & M \arrow[d, equal] \\
M\otimes(M\otimes M) \arrow[r, "\alpha_{M,M,M}"] \arrow[d, "f\otimes(f\otimes f)"'] \arrow[dr, phantom, "="] & (M\otimes M)\otimes M \arrow[r, "p\otimes M"] \arrow[d, "(f\otimes f)\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_2}\otimes 1_f}"] & M\otimes M \arrow[r, "p"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_2}}"] & M \arrow[d, "f"] \\
M'\otimes(M'\otimes M') \arrow[r, "\alpha_{M',M',M}"'] & (M'\otimes M')\otimes M' \arrow[r, "p'\otimes M'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}
=
$$

$$
=\quad
\begin{tikzcd}
M\otimes(M\otimes M) \arrow[rr, "M\otimes p"] \arrow[d, "f\otimes(f\otimes f)"'] \arrow[drr, phantom, "="] & & M\otimes M \arrow[r, "p"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_2}}"] & M \arrow[d, "f"] \\
M'\otimes(M'\otimes M') \arrow[rr, "M'\otimes p'"] \arrow[d, equal] & \arrow[d, phantom, "\xRightarrow{\alpha^{M'}}"] & M'\otimes M' \arrow[r, "p"'] & M \arrow[d, equal] \\
M'\otimes(M'\otimes M') \arrow[r, "\alpha_{M',M',M'}"'] & (M'\otimes M')\otimes M' \arrow[r, "p'\otimes M'"'] & M'\otimes M' \arrow[r, "p'"'] & M'
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
M \arrow[r, "\lambda_M^{-1}"] \arrow[d, equal] & I\otimes M \arrow[r, "u\otimes M"] \arrow[d, phantom, "\xRightarrow{\lambda^M}"] & M\otimes M \arrow[r, "p"] & M \arrow[d, equal] \\
M \arrow[rrr, "\mathrm{id}_M"'] \arrow[d, "f"'] & & \arrow[d, phantom, "="] & M \arrow[d, "f"] \\
M' \arrow[rrr, "\mathrm{id}_M"'] & & & M'
\end{tikzcd}
=
\begin{tikzcd}
M \arrow[r, "\lambda_M^{-1}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & I\otimes M \arrow[r, "u\otimes M"] \arrow[d, "I\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_0}\otimes 1_f}"] & M\otimes M \arrow[r, "p"] \arrow[d, "f\otimes f"'] \arrow[dr, phantom, "\xRightarrow{\overline{f_2}}"] & M \arrow[d, "f"] \\
M' \arrow[r, "\lambda_{M'}^{-1}"'] \arrow[d, equal] & I\otimes M' \arrow[r, "u'\otimes M'"'] \arrow[d, phantom, "\xRightarrow{\lambda^{M'}}"] & M'\otimes M' \arrow[r, "p'"'] & M' \arrow[d, equal] \\
M' \arrow[rrr, "\mathrm{id}_{M'}"'] & & & M'
\end{tikzcd}
$$

hold and the same goes for the one related to ρ^M, $\rho^{M'}$.

Note that these equations say precisely that $(f,\overline{f_0},\overline{f_2})$ is a lax monoidal morphism between (pre-)pseudomonoids $(M,p,u,\alpha^M,\lambda^M,\rho^M)$ and $(M',p',u',\alpha^{M'},\lambda^{M'},\rho^{M'})$, thus we already have the correct 1-cells in T_1-Alg_L.

The 2-functor T_1-$\mathsf{Alg}_L \to F$-Alg_L is fully faithful on 2-cells: a priori we need to impose the equation

$$
l'\cdot\psi_1\left(\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \end{tikzcd}\right) = \psi_2\left(\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\phi"] & \bullet \end{tikzcd}\right)\cdot l,
$$

but both ψ_1 and ψ_2 act as the identity on 2-cells and whiskering with l,l' does not affect the 2-cell because l,l' have identities as 1-cell components.

It follows that we already have the correct 2-cells in T_1-Alg as well. Since $T(F)\to T_1$ codetects identities and isomorphisms, the pseudo/strict T_1-morphisms are the $(f,\overline{f_0},\overline{f_2})$ s.t. $\overline{f_0},\overline{f_2}$ are invertible/identities.

Our T_1-Alg_L contains the 2-category of pseudomonoids and lax monoidal morphisms as a full 2-subcategory on those objects, for which the pentagon and unit triangle laws hold. We can use an equifier to describe this full 2-subcategory.

For this we consider a new 2-endofunctor $H\colon \mathcal{K} \to \mathcal{K}$ which sends M to $M \otimes (M \otimes (M \otimes M)) + M \otimes M$. We construct a 2-functor $\kappa_1\colon T_1\text{-}\mathsf{Alg}_L \to H\text{-}\mathsf{Alg}_L$ by sending (M, p, u) to

$$M \otimes (M \otimes (M \otimes M)) \xrightarrow{M \otimes (M \otimes p)} M \otimes (M \otimes M) \xrightarrow{M \otimes p} M \otimes M \xrightarrow{p} M,$$

$$M \otimes M \xrightarrow{M \otimes \lambda_M^{-1}} M \otimes (I \otimes M) \xrightarrow{M \otimes p} M \otimes M \xrightarrow{p} M$$

and a 2-functor $\kappa_2\colon T_1\text{-}\mathsf{Alg}_L \to H\text{-}\mathsf{Alg}_L$ by sending (M, p, u) to

$$M{\otimes}(M{\otimes}(M{\otimes}M)) \xrightarrow{\alpha_{M,M,M\otimes M}} (M{\otimes}M){\otimes}(M{\otimes}M) \xrightarrow{\alpha_{M\otimes M,M,M}} ((M{\otimes}M){\otimes}M){\otimes}M \xrightarrow{(p\otimes M)\otimes M} (M{\otimes}M){\otimes}M \xrightarrow{p\otimes M} M{\otimes}M \xrightarrow{p} M$$

$$M \otimes M \xrightarrow{\rho_M^{-1}} (M \otimes I) \otimes M \xrightarrow{(M\otimes u)\otimes M} (M \otimes M) \otimes M \xrightarrow{p\otimes M} M \otimes M \xrightarrow{p} M.$$

We extend this to 1-cells using the evident pastings of $\overline{f_0}$ and $\overline{f_2}$ and we let both 2-functors act as the identity on 2-cells.

Both restrict to 2-functors on strict morphisms, so by our general results they are induced by 2-monad morphisms

$$\begin{tikzcd} T(H) \arrow[r, bend left, "\hat{\kappa_1}"] \arrow[r, bend right, "\hat{\kappa_2}"'] & T_1 \end{tikzcd}$$

There are two ways of changing brackets in a word of four letters and they correspond to the two composites in MacLane's pentagon law. These and the cells in the unit triangle induce 2-cells $\beta_1, \beta_2\colon \kappa_1 \Rightarrow \kappa_2$ in $2-\mathbf{CAT}/\mathcal{K}$. We shall explain this for the associator and leave the unit law as an exercise. To make things more readable, we will simply write the tensor product in $\mathcal{K}$ as a concatenation, i.e. $M \otimes M$ will be MM. We construct two 2-natural transformations $\beta_1, \beta_2\colon \kappa_1 \to \kappa_2$ on 2-$\mathbf{Cat}/\mathcal{K}$ with component at $(M, p, u, \alpha, \lambda, \rho) \in T_1\text{-}\mathsf{Alg}_l$ resp. given by

$$\begin{tikzcd}
M(M(MM)) \arrow[r, "M(Mp)"] \arrow[dd, equal] \arrow[dr, phantom, "="] & M(MM) \arrow[rr, "Mp"] \arrow[d, equal] & & MM \arrow[rr, "p"] \arrow[d, phantom, "\Rightarrow \alpha^M"] & & M \arrow[dd, equal] \\
& M(MM) \arrow[dr, "\alpha"] \arrow[d, phantom, "="] & & {} & & \\
M(M(MM)) \arrow[ur, "M(Mp)"'] \arrow[dr, "\alpha"'] & {} & (MM)M \arrow[r, "pM"] \arrow[d, phantom, "="] & MM \arrow[rr, "p"] & & M \arrow[dd, equal] \\
& (MM)(MM) \arrow[ur, "(MM)p"] \arrow[rr, "p(MM)"] \arrow[d, "\alpha"] & {} \arrow[d, phantom, "="] & M(MM) \arrow[u, "Mp"'] \arrow[d, "\alpha"] \arrow[r, phantom, "\Rightarrow \alpha^M"] & {} & \\
& ((MM)M)M \arrow[rr, "(pM)M"'] & {} & (MM)M \arrow[r, "pM"'] & MM \arrow[r, "p"'] & M
\end{tikzcd}$$

and

$$\begin{tikzcd}
M(M(MM)) \arrow[r, "M(Mp)"] \arrow[d, equal] & M(MM) \arrow[rr, "Mp"] \arrow[dr, phantom, "\Rightarrow M\alpha^M"] & & MM \arrow[rr, "p"] \arrow[d, equal] & & M \arrow[d, equal] \\
M(M(MM)) \arrow[r, "M\alpha"'] & M((MM)M) \arrow[r, "M(pM)"'] \arrow[d, "\alpha"'] & M(MM) \arrow[r, "Mp"'] \arrow[d, "\alpha"] & MM \arrow[rr, "p"'] \arrow[dr, phantom, "\Rightarrow \alpha^M"] & & M \arrow[d, equal] \\
& (M(MM))M \arrow[r, "(Mp)M"] \arrow[d, equal] & (MM)M \arrow[rr, "pM"] \arrow[dr, phantom, "\Rightarrow \alpha^M M"] & & MM \arrow[r, "p"] \arrow[d, equal] & M \arrow[d, equal] \\
& (M(MM))M \arrow[r, "\alpha M"'] & ((MM)M)M \arrow[r, "(pM)M"'] & (MM)M \arrow[r, "pM"'] & MM \arrow[r, "p"'] & M
\end{tikzcd}$$

which has the correct codomain since the pentagon law holds in $\mathcal{K}$. In **Cat** these correspond precisely to the two composites in the pentagon law (involving two respectively three instances of the associator). A similar construction allows us to translate the unit axiom into two diagrams involving the second component of κ_1, κ_2 (exercise). These β_i are 2-natural since they are built from 2-natural transformations in $\mathcal{K}$ on 2-cells α, λ, ρ which are by definition compatible with all $(f, \overline{f_0}, \overline{f_2})$ in $T_1\text{-}\mathsf{Alg}_l$. Now we use the Proposition ensuring that $(-)\text{-}\mathsf{Alg}_l$ is fully faithful on 2-cells: the β_i are $(\hat{\beta}_i)^*$ for unique monad modifications $\widehat{\beta_i}\colon \widehat{\kappa_2} \Rrightarrow \widehat{\kappa_1}$. Let **PsMon** be the coequifier

$$\begin{tikzcd}
T(H) \arrow[r, bend left, "\widehat{\kappa_1}"] \arrow[r, bend right, "\widehat{\kappa_2}"'] \arrow[r, phantom, "\Downarrow\widehat{\beta_1}\ \Downarrow\widehat{\beta_2}"] & T_1 \arrow[r] & \mathbf{PsMon}
\end{tikzcd}$$

in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})$. Then $\mathbf{PsMon}\text{-}\mathsf{Alg}_l$ is the equifier of β_1 and β_2, so it is the full sub-2-category of $T_1\text{-}\mathsf{Alg}_l$ consisting of objects where β_1 and β_2 agree. Similarly for the unit law. Since an equifier does not affect 1- and 2-cells, our previous work shows that $\mathbf{PsMon}\text{-}\mathsf{Alg}_l$ is isomorphic to the 2-category of pseudomonoids, lax monoidal morphisms (in the usual sense) and monoidal 2-cells. We have also shown that $\mathbf{PsMon}\text{-}\mathsf{Alg}_p$ has as 1-cells the strong monoidal morphisms and $\mathbf{PsMon}\text{-}\mathsf{Alg}_l$ has as 1-cells the strict monoidal morphisms.

Our next example concerns categories with colimits of a given shape. This construction only works for conical colimits and only if the forgetful functor $V\colon \mathcal{V} \to \mathbf{Set}$ is conservative (e.g. $\mathbf{Set}, \mathbf{Mod}_R$ but not $\mathbf{sSet}, \mathbf{dgMod}_R, \mathbf{Cat}$). We also assume that $\mathcal{V}$ is a lfp cosmos so that $\mathcal{V}\text{-}\mathbf{Cat}$ is a lfp 2-category.

Let $\mathcal{D}$ be a κ-presentable (ordinary) category. We will show that the 2-category of small $\mathcal{V}$-categories with chosen $\mathcal{D}$-colimits and $\mathcal{V}$-functors which preserve $\mathcal{D}$-colimits is $T_\mathcal{D}\text{-}\mathsf{Alg}_p$ for a suitable κ-accessible 2-monad $T_\mathcal{D}$ on $\mathcal{V}\text{-}\mathbf{Cat}$.

Our assumptions imply that $\mathcal{C} \in \mathcal{V}\text{-}\mathbf{Cat}$ has chosen $\mathcal{D}$-colimits iff the diagonal $\mathcal{V}$-functor $\Delta\colon \mathcal{C} \to [\mathcal{D}, \mathcal{C}]$ has a (chosen) left adjoint.

So we start with the free 2-monad on the κ-accessible endo-2-functor $F := [\mathcal{D}, -]$. The objects of $F\text{-}\mathsf{Alg}_l$ already have a 1-cell $l\colon [\mathcal{D}, \mathcal{C}] \to \mathcal{C}$. We need to *insert* a unit and a counit and impose the triangle identities using an equifier.

There is a slight problem: note that the unit goes from $\mathrm{id}_{[\mathcal{D},\mathcal{C}]} \Rightarrow \Delta l$, so a priori this is a 2-cell $FC \rightrightarrows FC$ and doesn't need to live in $H\text{-}\mathsf{Alg}_l$. But F is a right 2-adjoint, so we can find a suitable H, namely $H = [\mathcal{D}, -] \otimes \mathcal{D}$: to give

$$\begin{tikzcd}
{[\mathcal{D},\mathcal{C}]} \arrow[r, bend left, "\mathrm{id}"] \arrow[r, bend right, "\Delta l"'] \arrow[r, phantom, "\Downarrow\eta"] & {[\mathcal{D},\mathcal{C}]}
\end{tikzcd}$$

is equivalent to giving

$$\begin{tikzcd}
{[\mathcal{D},\mathcal{C}] \otimes \mathcal{D}} \arrow[r, bend left, "(\mathrm{id})^{\#}"] \arrow[r, bend right, "(\Delta l)^{\#}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{C}
\end{tikzcd}$$

in $\mathcal{V}$-**Cat**. Thus our second endo-2-functor G sends $\mathcal{C}$ to $\mathcal{C} + [\mathcal{D}, \mathcal{C}] \otimes \mathcal{D}$ (the first term being for the counit). We form the inserter of the two 2-functors $F\text{-}\mathsf{Alg}_l \to G\text{-}\mathsf{Alg}_l$ sending $(\mathcal{C}, l\colon [\mathcal{D}, \mathcal{C}] \to \mathcal{C})$ to $(l\Delta\colon \mathcal{C} \to \mathcal{C}, \mathrm{id}^{\#}\colon [\mathcal{D}, \mathcal{C}] \otimes \mathcal{D} \to \mathcal{C})$ resp. $(\mathrm{id}\colon \mathcal{C} \to \mathcal{C}, (\Delta l)^{\#}\colon [\mathcal{D}, \mathcal{C}] \otimes \mathcal{D} \to \mathcal{C})$. Here we really need to be able to give in non-invertible 2-cells. The 1-cells in $F\text{-}\mathsf{Alg}_l$ are pairs (F, λ) consisting of a $\mathcal{V}$-functor $f\colon \mathcal{C} \to \mathcal{C}'$ and a 2-cell

$$\begin{tikzcd}
{[\mathcal{D},\mathcal{C}]} \arrow[r, "l"] \arrow[d, "{[\mathcal{D},f]}"'] \arrow[dr, phantom, "\xRightarrow[\lambda]{}"] & \mathcal{C} \arrow[d, "f"] \\
{[\mathcal{D},\mathcal{C}']} \arrow[r, "l'"'] & \mathcal{C}'
\end{tikzcd}$$

and the two 2-functors send this to

$$\left(\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & {[\mathcal{D},\mathcal{C}]} \arrow[r, "l"] \arrow[d, dashed, "{[\mathcal{D},f]}"] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{C}' \arrow[r, "\Delta"'] & {[\mathcal{D},\mathcal{C}']} \arrow[r, "l'"'] & \mathcal{C}'
\end{tikzcd}, \; 1_{\mathrm{id}^{\#}}\right)$$

and

$$\left(1_{\mathrm{id}_{\mathcal{C}}}, \left(\begin{tikzcd}
{[\mathcal{D},\mathcal{C}]} \arrow[r, "l"] \arrow[d, "{[\mathcal{D},f]}"'] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & \mathcal{C} \arrow[r, "\Delta"] \arrow[d, "f"] \arrow[dr, phantom, "="] & {[\mathcal{D},\mathcal{C}]} \arrow[d, "{[\mathcal{D},f]}"] \\
{[\mathcal{D},\mathcal{C}']} \arrow[r, "l'"'] & \mathcal{C}' \arrow[r, "\Delta"'] & {[\mathcal{D},\mathcal{C}']}
\end{tikzcd}\right)^{\#}\right)$$

respectively. Both act as the identity on 2-cells. Using the adjunction $- \otimes \mathcal{D} \dashv [\mathcal{D}, -]$, we find that the coinserter T_1 of the resulting 2-monad morphism has $T_1\text{-}\mathsf{Alg}_l$ given by quadruples $(\mathcal{C}, l, \eta, \epsilon)$, where $\eta\colon \mathrm{id} \Rightarrow \Delta l, \epsilon\colon l\Delta \Rightarrow \mathrm{id}$ (subject to no axioms) and 1-cells are (f, λ) s.t.

$$\begin{tikzcd}
{[\mathcal{D},\mathcal{C}]} \arrow[r, "l"] \arrow[d, "{[\mathcal{D},f]}"'] \arrow[dr, phantom, "\xRightarrow[\lambda]{}"] & \mathcal{C} \arrow[r, "c \mapsto \Delta_c"] \arrow[d, "f"] \arrow[dr, phantom, "//"] & {[\mathcal{D},\mathcal{C}]} \arrow[d, "{[\mathcal{D},f]}"] \\
{[\mathcal{D},\mathcal{C}']} \arrow[r, "l'"'] & \mathcal{C}' \arrow[r, "c' \mapsto \Delta_{c'}"'] & {[\mathcal{D},\mathcal{C}']}
\end{tikzcd}
=
\begin{tikzcd}
& \mathcal{C} \arrow[dr, "\Delta"] & \\
{[\mathcal{D},\mathcal{C}]} \arrow[ur, "l"] \arrow[rr, "\mathrm{id}_{[\mathcal{D},\mathcal{C}]}"'] \arrow[d, "{[\mathcal{D},f]}"'] \arrow[urr, phantom, "\Uparrow\eta"] \arrow[drr, phantom, "="] & & {[\mathcal{D},\mathcal{C}]} \arrow[d, "{[\mathcal{D},f]}"] \\
{[\mathcal{D},\mathcal{C}']} \arrow[rr, "\mathrm{id}_{[\mathcal{D},\mathcal{C}']}"'] & & {[\mathcal{D},\mathcal{C}']}
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "c\mapsto\Delta_c"] \arrow[rr, bend left, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & {[\mathcal{D},\mathcal{C}]} \arrow[r, "l"] \arrow[d, "{[\mathcal{D},f]}"'] \arrow[u, phantom, "\Nearrow\epsilon"] \arrow[dr, phantom, "\xRightarrow{\lambda}"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{C}' \arrow[r, "c'\mapsto\Delta_{c'}"'] & {[\mathcal{D},\mathcal{C}']} \arrow[r, "l'"'] & \mathcal{C}'
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{C} \arrow[rr, "\mathrm{id}_{\mathcal{C}}"] \arrow[d, "f"'] \arrow[drr, phantom, "="] & & \mathcal{C} \arrow[d, "f"] \\
\mathcal{C}' \arrow[rr, "\mathrm{id}_{\mathcal{C}'}"] \arrow[dr, "c'\mapsto\Delta_{c'}"'] & \Nearrow\epsilon & \mathcal{C}' \\
& {[\mathcal{D},\mathcal{C}]} \arrow[ur, "l'"'] &
\end{tikzcd}
$$

We now impose the triangle identities using an equifier in the same manner as before (using the necessary 2-adjunction for the one, where the target is not $\mathcal{C}$). This is isomorphic to $T_{\mathcal{D}}\text{-}\mathsf{Alg}_l$ where $T_{\mathcal{D}}$ denotes the corresponding coequifier in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{V}\text{-}\mathbf{Cat})$. Since this is a coequifier, the 1-cells and 2-cells are the same as in $T_1\text{-}\mathsf{Alg}_l$. However, now $l \dashv \Delta$ with unit η and counit ϵ, so the above coequifier say that λ is the *mate* of $1_{\mathcal{C}}$. So each f has a *unique* lax morphism structure. The pseudo T_1-morphism are the ones where λ is invertible, so the same is true for $T_{\mathcal{D}}$. The components of λ are precisely the colimit comparison morphisms, so the pseudo $T_{\mathcal{D}}$-morphisms are exactly the $\mathcal{D}$-colimits preserving $\mathcal{V}$-functors. One can also check that this works for 2-cells, meaning all $\mathcal{V}$-natural transformations are $T_{\mathcal{D}}$-transformations. In T_1 there is a condition which becomes automatic when λ is the mate of $1_{\mathcal{C}}$.

Remark 0.2.26. The free objects for $T_{\mathcal{D}}$ should correspond to the $\mathcal{D}$-colimits closure in the diagram category $[\mathcal{C},\mathcal{V}]$ of the representables. For this we need to understand "how free" $T_{\mathcal{D}}(\mathcal{C})$ actually is in $T_{\mathcal{D}}\text{-}\mathsf{Alg}_p$ (as opposed to $T_{\mathcal{D}}\text{-}\mathsf{Alg}_s$).

Remark 0.2.27. If we want to get the 2-monad for categories with colimits of shape $\{\mathcal{D}_i\}_{i\in I}$ for some set of ordinary categories, we simply take the coproduct $\coprod T_{\mathcal{D}_i}$ in $2\text{-}\mathsf{Mnd}(\mathcal{V}\text{-}\mathbf{Cat})$ (all $\mathcal{D}_i$ are κ-presentable). E.g. given shapes for binary coproducts, initial object and coequalizers we get finitely cocomplete categories in the case $\mathcal{V} = \mathbf{Set}$. Our final example concerns 2-categories of 2-functors. Let $\mathcal{K}$ be a cocomplete 2-category and $\mathcal{A}$ a small 2-category. Then $[\mathcal{A},\mathcal{K}]$, the 2-category of (strict) 2-functors, (strict) 2-natural transformations and modifications is the 2-category of algebras for the 2-monad

$$
\begin{aligned}
T\colon [\mathrm{Ob}\,\mathcal{A},\mathcal{K}] &\longrightarrow [\mathrm{Ob}\,\mathcal{A},\mathcal{K}] \\
(X_a)_{a\in\mathcal{A}} &\mapsto \left(\sum_{a\in\mathcal{A}} \mathcal{A}(a,b)\odot X_a\right)_{b\in\mathcal{A}}
\end{aligned}
$$

by definition in our case if $\mathcal{K} = \mathbf{Cat}$, and in general it follows from the adjunction defining the copower:

$$(\mathcal{A}(a,b)\odot X_a \to X_b) \leftrightsquigarrow (\mathcal{A}(a,b) \to \mathcal{K}(X_a,X_b)).$$

The coproduct of T^2 at $c\in\mathcal{A}$ is

$$
\begin{aligned}
\left(T^2(X_a)_{a\in\mathcal{A}}\right)_c &= \sum_b \mathcal{A}(b,c)\odot (T(X_a)_{a\in\mathcal{A}})_b \\
&= \sum_b \mathcal{A}(b,c)\odot\left(\sum_a \mathcal{A}(a,b)\odot X_a\right) \\
&\cong \sum_{a,b} (\mathcal{A}(b,c)\times\mathcal{A}(a,b))\odot X_a.
\end{aligned}
$$

The unit and multiplication are given by the identities resp. composition in $\mathcal{A}$. To give a lax T-morphism $(F_a)_{a\in\mathcal{A}} \to (G_a)_{a\in\mathcal{A}}$ amounts to giving a pair $(f, \overline{f})$ whose f is simply a morphism of collections, i.e. a 1-cell $f_a\colon F_a \to G_a$ for each $a \in \mathcal{A}$ and $\overline{f}$ is a 2-cell

$$\begin{tikzcd}
\sum_a \mathcal{A}(a,b)\odot F_a \arrow[r, "\varphi_b"] \arrow[d, "\sum_a \mathcal{A}(a,b)\odot f_a"'] \arrow[dr, phantom, "\overset{}{\underset{\overline{f_b}}{\Longrightarrow}}"] & F_b \arrow[d, "f_b"] \\
\sum_a \mathcal{A}(a,b)\odot G_a \arrow[r, "\gamma_b"'] & G_b
\end{tikzcd}$$

for each $b \in \mathcal{A}$. Here φ and γ encode the algebraic structure of F and G. By the universal property of coproducts, to give $\overline{f_b}$ is equivalent to giving a 2-cell for each component $a \in \mathcal{A}$, which by universal property of copower corresponds to a 2-cell

$$\begin{tikzcd}
\mathcal{A}(a,b) \arrow[r, "F_{a,b}"] \arrow[d, "G_{a,b}"'] \arrow[dr, phantom, "\overset{\overline{f_{a,b}}}{\Longrightarrow}"] & \mathcal{K}(F_a,F_b) \arrow[d, "\mathcal{K}(F_a,f_b)"] \\
\mathcal{K}(G_a,G_b) \arrow[r, "\mathcal{K}(f_a,G_b)"'] & \mathcal{K}(F_a,G_b)
\end{tikzcd}$$

in **Cat**. So this is simply a natural transformation in **Cat**, which has components

$$\begin{tikzcd}
F_a \arrow[r, "F_\psi"] \arrow[d, "f_a"'] \arrow[dr, phantom, "\underset{f_\psi}{\Longrightarrow}"] & F_b \arrow[d, "f_b"] \\
G_a \arrow[r, "G_\psi"'] & G_b
\end{tikzcd}$$

for each $\psi\colon a \to b$ in $\mathcal{A}$. So the data of a lax T-morphism corresponds bijectively to the data of a lax natural transformation $F \Rightarrow G$. In fact, $(f, \overline{f})$ satisfies the axioms of a lax T-morphism if and only if (f_a, f_ψ) form a lax natural transformation. The naturality of $\overline{f_{a,b}}$ is precisely the compatibility of f_ψ with 2-cells and the two axioms for a T-morphism correspond to the pasting and identity axioms for a lax natural transformation. This follows since the axioms for T-morphisms can be checked componentwise. Similarly, one can check that T-transformations are the modifications. Finally, a 2-cell out of a coproduct is invertible if and only if its components are and

$$\begin{tikzcd}
\mathcal{A}(a,b)\odot X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Y
\end{tikzcd}$$

is an isomorphism if and only if

$$\begin{tikzcd}
\mathcal{A}(a,b) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathcal{K}(X,Y)
\end{tikzcd}$$

is an isomorphism, so the pseudo T-morphism are precisely the (f_a, f_ψ) s.t. each f_ψ is an isomorphism. Thus the pseudo T-morphisms are precisely the pseudonatural transformations.

0.3 Limits and colimits in T-Alg_p

Recall that for a 1-monad T on a complete category, T-Alg is always complete. The enriched version of this also works. In particular, T-Alg_s is complete if $\mathcal{K}$ is. What about T-Alg_p and T-Alg_l? We start with some positive results.

Proposition 0.3.1. If $\mathcal{K}$ has products and $T\colon \mathcal{K} \to \mathcal{K}$ is a 2-monad, then the products in $T\text{-}\mathsf{Alg}_s$ are products in $T\text{-}\mathsf{Alg}_p$.

Proof. We already know that products exist in $T\text{-}\mathsf{Alg}_s$, so this amounts to checking the universal property. This is similar to the case we saw involving the colax limit of an arrow. In the next few propositions we will see more examples of this kind, so we have this as an exercise. $\square$

Remark 0.3.2. We actually only proved existence of products in $T\text{-}\mathsf{Alg}_s$ if $\mathcal{K}$ is complete. It is true in general if $\mathcal{K}$ has products (exercise). The same remains true in the following propositions.

Proposition 0.3.3. If $\mathcal{K}$ has (iso-)inserters, then $T\text{-}\mathsf{Alg}_p$ has (iso-)inserters. The universal 1-cell is a strict T-morphism and it detects strict T-morphisms.

Proof. We do the inserter case; the iso-inserter is similar. Let $(f,\overline{f}),(g,\overline{g})\colon (A,a) \rightsquigarrow (B,b)$ be two pseudo T-morphisms and let

$$\begin{tikzcd}
 & A \arrow[dr, "f"] \arrow[dd, phantom, "\Downarrow\lambda"] & \\
I \arrow[ur, "p"] \arrow[dr, "p"'] & & B \\
 & A \arrow[ur, "g"'] &
\end{tikzcd}$$

be the inserter in $\mathcal{K}$. We have $a \cdot Tp\colon TI \to A$ and a 2-cell

$$fa \cdot Tp \xRightarrow{\overline{f}^{-1}\cdot Tp} b \cdot Tf \cdot Tp \xRightarrow{b\cdot T\lambda} b \cdot Tg \cdot Tp \xRightarrow{\overline{g}\cdot Tp} ga \cdot Tp$$

and so from the universal property we get a unique $i\colon TI \to I$ s.t. $p \cdot i = a \cdot Tp$ and the equation

$$\begin{tikzcd}
 & TA \arrow[dr, "Tf"] \arrow[dd, "a"] & \\
TI \arrow[ur, "Tp"] \arrow[dd, "i"'] & \arrow[d, phantom, "\overline{f}\Downarrow"] & TB \arrow[dd, "b"] \\
 & A \arrow[dr, "f"] \arrow[dd, phantom, "\lambda\Downarrow"] & \\
I \arrow[ur, "p"] \arrow[dr, "p"'] & & B \\
 & A \arrow[ur, "g"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TA \arrow[dr, "Tf"] \arrow[dd, phantom, "\Downarrow\lambda"] & \\
TI \arrow[ur, "Tp"] \arrow[dr, "Tp"'] \arrow[dd, "i"'] & & TB \arrow[dd, "b"] \\
 & TA \arrow[ur, "Tg"'] \arrow[dd, "a"] & \arrow[d, phantom, "\overline{g}\Downarrow"] \\
I \arrow[dr, "p"'] & & B \\
 & A \arrow[ur, "g"'] &
\end{tikzcd}$$

holds. As in the proof of the "colax limit of an arrow", we use the axioms for $(f,\overline{f})$ and $(g,\overline{g})$ and the 2-naturality of η and μ to show that (I,i) is a T-algebra. By construction, $p\colon (I,i) \to (A,a)$ is a strict T-morphism and λ a T-transformation. It remains to check the universal property, so consider

$$\begin{tikzcd}
 & (A,a) \arrow[dr, squiggly] \arrow[dd, phantom, "\Downarrow\mu"] & \\
(X,x) \arrow[ur, squiggly, "(q,\overline{q})"] \arrow[dr, squiggly, "(q,\overline{q})"'] & & (B,b) \\
 & (A,a) \arrow[ur, squiggly] &
\end{tikzcd}$$

in T-Alg_p. This means that the equation

$$
\begin{tikzcd}
 & & TA \arrow[drr, "Tf"] \arrow[dd, "a"] & & \\
TX \arrow[urr, "Tq"] \arrow[dd, "x"'] & \arrow[d, phantom, "\Downarrow\overline{q}"] & & \arrow[d, phantom, "\overline{f}\Downarrow"] & TB \arrow[dd, "b"] \\
 & & A \arrow[drr, "f"] \arrow[dd, phantom, "\mu\Downarrow"] & & \\
X \arrow[urr, "q"] \arrow[drr, "q"'] & & & & B \\
 & & A \arrow[urr, "g"'] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & & TA \arrow[drr, "Tf"] \arrow[dd, phantom, "\Downarrow T\mu"] & & \\
TX \arrow[urr, "Tq"] \arrow[drr, "Tq"] \arrow[dd, "x"'] & & & & TB \arrow[dd, "b"] \\
 & \arrow[d, phantom, "\overline{q}\Downarrow"] & TA \arrow[urr, "Tg"'] \arrow[dd, "a"] & \arrow[d, phantom, "\overline{g}\Downarrow"] & \\
X \arrow[drr, "q"'] & & & & B \\
 & & A \arrow[urr, "g"'] & &
\end{tikzcd}
\qquad (*)
$$

holds. We have a unique 1-cell $h\colon X \to I$ s.t. $ph = q$ and $\lambda h = \mu$ from the universal property of (I, p, λ) in $\mathcal{K}$. Thus $\overline{q}$ can be seen as a 2-cell

$$p \cdot i \cdot Th = a \cdot Tp \cdot Th = a \cdot Tq \overset{\overline{q}}{\Rightarrow} q \cdot x = p \cdot h \cdot x$$

in $\mathcal{K}$. Plugging this into $(*)$ and using $ph = q, \lambda h = \mu$, we find that $(\lambda hx) \cdot (f\overline{q}) \cdot (\overline{f} \cdot Tp \cdot Th) = (g\overline{q}) \cdot (\overline{g} \cdot Tp \cdot Th) \cdot (b \cdot T\lambda \cdot Th)$ holds. Using the definition of i in terms of $\overline{f}^{-1}$, we find that the equation holds if and only if $(\lambda \cdot h \cdot x) \cdot (f \cdot \overline{q}) = (g \cdot \overline{q}) \cdot (\lambda \cdot i \cdot Th)$ holds. From the 2-dimensionality of the universal property of (I, p, λ) it follows that there exists a unique $\overline{h}\colon i \cdot Th \Rightarrow x \cdot h$ s.t. $p\overline{h} = \overline{q}$. Using the uniqueness part of the 2-dimensional universal property plus the fact that $(q, \overline{q})$ is a pseudo T-morphism, it follows that $(h, \overline{h})$ is a pseudo T-morphism. This $(h, \overline{h})$ is clearly the unique 1-cell with $p \cdot (h, \overline{h}) = (q, \overline{q})$, so this shows the 1-dimensional universal property. Checking the 2-dimensional universal property is left as an exercise. $\square$

We also have the following statement similar to the previous one.

Proposition 0.3.4. Let $\mathcal{K}$ be a 2-category with equifiers. Then T-Alg_P has equifiers, which are preserved by U_P. The universal 1-cell is a strict T-morphism detecting strict T-morphisms.

Proof. Consider a pair of 2-cells in T-Alg_P

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, squiggly, bend left, "f"] \arrow[r, squiggly, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & \mathcal{B}
\end{tikzcd},
$$

with equifier $p\colon E \to A$ in $\mathcal{K}$. We have to define an algebra structure on E and check the universal property.

The T-transformation axiom for α says that $(\alpha \cdot a) \cdot \overline{f} = \overline{g} \cdot (b \cdot T\alpha)$, so $((\alpha \cdot a) \cdot Tp)(\overline{f} \cdot Tp = (\overline{g} \cdot Tp) \cdot (b \cdot T\alpha \cdot Tp)$ holds and similarly with β in place of α. Since $\overline{f}$ is an isomorphism, this implies that $\alpha \cdot a \cdot Tp = \beta \cdot a \cdot Tp$. It follows that there exists a unique $e\colon TE \to E$ s.t. $p \cdot e = a \cdot Tp$.

As in the other cases, one checks that (E, e) is a T-algebra with the desired universal property. Note that p is a strict T-morphism by construction. The claim about detecting strict morphisms is also left as an exercise. $\square$

We have shown that T-Alg_P has products, inserters, equifiers, or is PIE-limits for short (namely anything that can be built from there).

Remark 0.3.5. In general, T-Alg_P does not have equalizers.

Example 0.3.6. Consider the 2-category of small categories with an initial object and functors preserving it up to isomorphism. This is T-Alg_P for a suitable T. We can show that the equalizer of $0, 1\colon * \to \{0 \cong 1\}$ doesn't exist. Indeed, if $E \to *$ were the equalizer, then the unique object can't be in the image since it is mapped to two different objects in $\{0 \cong 1\}$. It follows that the image is $\emptyset$ and therefore $E = \emptyset$, which does not have an initial object .

A consequence of this remark is that, even if $\mathcal{K}$ is complete, T-Alg_P will in general not be complete. However, we will see that PIE-limits can be used to construct all bilimits, i.e. weak 2-limits.

Remark 0.3.7. The situation in T-Alg_L is even worse: it does have products, but it has neither inserters nor equifiers.

Example 0.3.8. Consider the 2-category of finitely cocomplete small categories with all functors. This is again T-Alg_L for a suitable (finitary) T. Since it has products, if it had inserters it would also have comma objects, but the comma object

$$\begin{tikzcd}
\mathbf{Set}(X,Y) \arrow[r] \arrow[d] & * \arrow[d, "c_X"] \arrow[dl, phantom, "\swarrow"] \\
* \arrow[r, "c_Y"'] & \mathbf{Set}
\end{tikzcd}$$

is the discrete category $\mathbf{Set}(X,Y)$ in $\mathbf{Cat}$, so if the comma object in T-Alg_L existed it would induce a functor to $\mathbf{Set}(X,Y)$. This is however impossible if $\mathbf{Set}(X,Y)$ has more than one object, since it contradicts the existence of an initial object. More precisely, the existence of an initial object implies that the unique morphism from the comma object in T-Alg_L to $\mathbf{Set}(X,Y)$ factors through the inclusion $\{f\} \to \mathbf{Set}(X,Y)$ for some $f\colon X \to Y$. But in T-Alg_L we also have the 2-cell

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[d, "X"] \arrow[dl, phantom, "\swarrow \tilde{g}"] \\
* \arrow[r, "Y"'] & \mathbf{Set}
\end{tikzcd}$$

for any $\tilde{g}\colon X \to Y$, so g would also have to lie in the image of the comparison morphism. It follows that we don't have products.

We will also show that it lacks equifiers. Indeed, consider two distinct morphisms $f, g\colon X \to Y$. They give two 2-cells $c_f, c_g\colon c_X \to c_Y$ and the equifier in $\mathbf{Cat}$ is $\emptyset$, thus the equifier in T-Alg_L, if it existed, would map to $\emptyset$, hence it would be $\emptyset$ itself. However, $\emptyset$ is not finitely cocomplete, thus the equifier doesn't exist.

These facts emphasize that using $\overline{f}^{-1}$ was crucial in lifting inserters to algebras.

In order to investigate what kinds of limits we can build using products, inserters and equifiers and to study the existence of colimits in T-Alg_P, we need the notion of flexible algebras. This, in turn, requires the existence of (lax) pseudo morphism classifiers.

Definition 0.3.9. Let T be a 2-monad on $\mathcal{K}$ and A a T-algebra. We write $T\text{-}\mathsf{Alg}_S \xrightarrow{J} T\text{-}\mathsf{Alg}_P \xrightarrow{K} T\text{-}\mathsf{Alg}_L$ for the inclusions. A pseudo (respectively lax) morphism classifier is a representing object for $T\text{-}\mathsf{Alg}_P(A, J-)$ (respectively $T\text{-}\mathsf{Alg}_L(A, J-)$) in $T\text{-}\mathsf{Alg}_S$, that is an object QA (respectively $Q_L A$) with a pseudo T-morphism $h_A \colon A \rightsquigarrow QA$ (respectively a lax T-morphism $h^L_A \colon A \rightsquigarrow Q_L A$) s.t. the induced 2-natural transformation $T\text{-}\mathsf{Alg}_S(QA, B) \to T\text{-}\mathsf{Alg}_P(A, JB)$ (respectively $T\text{-}\mathsf{Alg}_S(Q_L A, B) \to T\text{-}\mathsf{Alg}_P(A, JB)$) is an isomorphism.

We are asking for each $A \rightsquigarrow B$ to factor uniquely through a strict T-morphism $Q \to B$ plus a 2-dimensional property.

Remark 0.3.10. If the pseudo morphism classifier exists for all A, then J has Q as a left 2-adjoint. Similarly, KJ has a left 2-adjoint if $Q^L A$ exists for all A. The object QA is often denoted A'.

Theorem 0.3.11 (Lack)**.** If $T\text{-}\mathsf{Alg}_L$ has codescent objects (respectively lax codescent objects), then it has (lax) pseudo morphism classifiers for all A.

Proof. We have to translate the data of a lax/pseudo T-morphism $(f, \overline{f}) \colon (A, a) \to (B, b)$ into a diagram in $T\text{-}\mathsf{Alg}_S$ and then take its colimit.

The 1-cell $f \colon A \to B$ in $\mathcal{K}$ corresponds bijectively to a 1-cell $g \colon TA \to B$ in $T\text{-}\mathsf{Alg}_S$. The bijection sends f to $g = b \cdot Tf$ since b is the counit of $T \dashv U_S$.

Giving a 2-cell

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\overset{\overline{f}}{\Longrightarrow}"] & A \arrow[d, "f"] \\ TB \arrow[r, "b"'] & B \end{tikzcd}$$

in $\mathcal{K}$ amounts to giving a 2-cell

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T^2f"'] \arrow[dr, phantom, "\overset{\xi}{\Swarrow}"] & TA \arrow[d, "Tf"] \\ T^2B \arrow[r, "\mu_B"'] & TB \end{tikzcd}$$

in $T\text{-}\mathsf{Alg}_S$: the codomain has to be $b \cdot T(f \cdot a) = b \cdot Tf \cdot Ta = g \cdot Ta$ and the domain is $b \cdot T(b \cdot Tf) = b \cdot Tb \cdot T^2 f = b \cdot \mu_B \cdot T^2 f = b \cdot Tf \cdot \mu_A = g \cdot \mu_A$.

The condition $\overline{f} \cdot \eta_A = 1_f$ becomes $\xi \cdot T\eta_A = 1_g$ while the other axiom becomes

$$\begin{tikzcd} & T^2A \arrow[rr, "\mu_A"] \arrow[dr, "Ta"] & & TA \arrow[dr, "g"] & \\ T^3A \arrow[ur, "\mu_{TA}"] \arrow[dr, "T^2a"'] & & TA \arrow[rr, "g"] \arrow[ur, phantom, "\overset{\xi}{\Swarrow}"] \arrow[dr, phantom, "\underset{\xi}{\Searrow}"] & & B \\ & T^2A \arrow[ur, "\mu_A"] \arrow[rr, "Ta"'] & & TA \arrow[ur, "g"'] & \end{tikzcd} = \begin{tikzcd} & T^2A \arrow[rr, "\mu_A"] & & TA \arrow[dr, "g"] \arrow[dd, phantom, "\xi\Downarrow"] & \\ T^3A \arrow[ur, "\mu_{TA}"] \arrow[dr, "T^2a"'] \arrow[rr, "T\mu_A"] & & T^2A \arrow[ur, "\mu_A"] \arrow[dr, "Ta"'] & & B \\ & T^2A \arrow[rr, "Ta"'] & & TA \arrow[ur, "g"'] & \end{tikzcd}$$

in $T\text{-}\mathsf{Alg}_S$. This is precisely a (lax) codescent datum on the truncated simplicial diagram

$$\begin{tikzcd} T^3A \arrow[r, shift left=2, "T^2A"] \arrow[r, "T\mu_A" description] \arrow[r, shift right=2, "\mu_{TA}"'] & T^2A \arrow[r, shift left=1, "Ta"] \arrow[r, shift right=1, "\mu_A"'] & TA \end{tikzcd}$$

and one can check that the 2-dimensional aspect of a codescent object corresponds precisely to the fact that 2-cells $g \Rightarrow g'$ compatible with ξ correspond bijectively to T-transformations $(f, \overline{f}) \Rightarrow (f', \overline{f'})$. $\square$

0.4 Codescent objects

To be moved elsewhere. This section fills a previous gap.

Definition 0.4.1. Consider a truncated simplicial diagram

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1" description] \arrow[r, shift right=2, "d_0"'] & X_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X_0 \arrow[l, "s_0" description] \end{tikzcd}$$

in a 2-category $\mathcal{K}$. A *codescent datum* in this diagram $X_\bullet$ is a pair (g, ξ) of a 1-cell $g\colon X_0 \to C$ and a 2-cell

$$\begin{tikzcd} & X_0 \arrow[dr, "g"] \arrow[dd, phantom, "\Downarrow \xi"] & \\ X_1 \arrow[ur, "d_1"] \arrow[dr, "d_0"'] & & C \\ & X_0 \arrow[ur, "g"'] & \end{tikzcd}$$

s.t. the axiom $\xi \cdot s_0 = 1_g$ and the equation

$$\begin{tikzcd} & X_1 \arrow[rr, "d_1"] \arrow[dr, "d_0"] & & X_0 \arrow[dr, "g"] & \\ X_2 \arrow[ur, "d_2"] \arrow[dr, "d_0"'] \arrow[rr, phantom, "="] & & X_0 \arrow[rr, "g"] \arrow[ur, phantom, "\Swarrow \xi"] \arrow[dr, phantom, "\Searrow \xi"] & & C \\ & X_1 \arrow[rr, "d_0"'] \arrow[ur, "d_1"] & & X_0 \arrow[ur, "g"'] & \end{tikzcd} = \begin{tikzcd} & X_1 \arrow[rr, "d_1"] & & X_0 \arrow[dr, "g"] \arrow[dd, phantom, "\Downarrow \xi"] & \\ X_2 \arrow[ur, "d_2"] \arrow[dr, "d_0"'] \arrow[rr, "d_1"] & & X_1 \arrow[ur, "d_1"] \arrow[dr, "d_0"'] & & C \\ & X_1 \arrow[rr, "d_0"'] & & X_0 \arrow[ur, "g"'] & \end{tikzcd}$$

holds.

A morphism of descent data (g, ξ) and (g', ξ') with the same target C is a 2-cell $\alpha\colon g \Rightarrow g'$ s.t.

$$\begin{tikzcd} & X_0 \arrow[dr, bend left, "g"] \arrow[dr, "g'"'] \arrow[dd, phantom, "\xi' \Downarrow"] & \\ X_1 \arrow[ur, "d_1"] \arrow[dr, "d_0"'] & & C \\ & X_0 \arrow[ur, "g'"'] & \end{tikzcd} = \begin{tikzcd} & X_0 \arrow[dr, "g"] \arrow[dd, phantom, "\xi \Downarrow"] & \\ X_1 \arrow[ur, "d_1"] \arrow[dr, "d_0"'] & & C \\ & X_0 \arrow[ur, "g"] \arrow[ur, bend right, "g'"'] & \end{tikzcd}$$

holds.

Sending $C \in \mathcal{K}$ to the category of descent data with target C defines a 2-functor $\mathcal{K} \to \mathbf{Cat}$. A codescent object for $X_\bullet$ is a representing object for the 2-functor. By Yoneda, this amounts to a universal such codescent datum.

An iso-codescent object is one where each ξ is invertible.

Remark 0.4.2. Codescent objects are weighted colimits and the weight is given simply by the inclusion $\overline{\Delta_{\leq 2}} \to \mathbf{Cat}$ (where the domain category has the objects $[0], [1], [2]$ and all of the arrows of Δ but the codegeneracies $[1] \to [2]$). The dual notion for cosimplicial objects is called descent objects.

We can also consider these for pseudofunctors $\overline{\Delta_{\leq 2}}^{\mathrm{op}} \to \mathcal{K}$. In this case, one has to replace the equalities above coming from the simplicial identities by the coherence isomorphisms, i.e.

$$
\begin{tikzcd}
 & & X_0 \arrow[dr, "g"] & \\
X_0 \arrow[r, dashed, "s_0"] \arrow[urr, equal, bend left] \arrow[drr, equal, bend right] & X_1 \arrow[ur, dashed, "d_1"'] \arrow[dr, dashed, "d_0"] \arrow[u, phantom, "\cong"] \arrow[d, phantom, "\cong"] & \Downarrow\xi & C \\
 & & X_0 \arrow[ur, "g"'] &
\end{tikzcd}
\qquad = \qquad 1_g \ .
$$

Remark 0.4.3. The terminology is not entirely standardized: sometimes the "iso" version is called the (co)descent object and the above is called a lax codescent object. For emphasis, it is perhaps best to always refer to iso-(co)descent objects and lax (co)descent objects.

Proposition 0.4.4. If $\mathcal{K}$ is a complete and cocomplete 2-category, then so is $T\text{-}\mathsf{Alg}_S$ and thus QA, Q_LA exist for all A.

Proof. We showed this for general $\mathcal{V}$ in the past course. □

As already mentioned, in this case Q defines a left 2-adjoint to $J\colon T\text{-}\mathsf{Alg}_S \to T\text{-}\mathsf{Alg}_P$. We write $e_A\colon QA \to A$ for the counit of the adjunction, so this is the unique strict T-morphism s.t. the triangle

$$
\begin{tikzcd}
A \arrow[r, rightsquigarrow, "h_A"] \arrow[dr, equal] & QA \arrow[d, "e_A"] \\
 & A
\end{tikzcd}
$$

commutes. Notice that this is just one of the triangle identities with J omitted from the notation.

Remark 0.4.5. Often the notation $h_A = q_A$ and $e_A = p_A$ is used instead.

What can we say about this adjunction?

Proposition 0.4.6. If $h_A\colon A \rightsquigarrow QA$ exists and the pseudolimit of h_A exists in $\mathcal{K}$, then e_A is right adjoint to h_A with identity unit and invertible counit $\rho_A\colon h_A \cdot e_A \Rightarrow \mathrm{id}_{QA}$. In particular, e_A and h_A are equivalences in $T\text{-}\mathsf{Alg}_P$.

Proof. Existence of the pseudolimit in $\mathcal{K}$ implies the existence in $T\text{-}\mathsf{Alg}_P$ of the pseudolimit of the form

$$
\begin{tikzcd}
 & C \arrow[dl, "u"'] \arrow[dr, "v"] & \\
A \arrow[rr, rightsquigarrow, "h_A"'] & & B
\end{tikzcd}
,
$$

i.e. u, v are strict, and this factorization s.t.

$$
\begin{tikzcd}
 & QA \arrow[d, rightsquigarrow, "r"] \arrow[ddl, bend right, "d"'] \arrow[ddr, rightsquigarrow, "h_A"] & \\
 & C \arrow[dl, "u"'] \arrow[dr, "v"] & \\
A \arrow[rr, rightsquigarrow, "h_A"'] & & QA
\end{tikzcd}
\qquad = \qquad
\begin{tikzcd}
 & A \arrow[dr, rightsquigarrow, "h_A"] \arrow[d, phantom, "="] & \\
A \arrow[ur, equal] \arrow[rr, rightsquigarrow, "h_A"'] & {} & QA
\end{tikzcd}
$$

in $T\text{-}\mathsf{Alg}_P$. By the universal property of QA, there exists a unique $w\colon QA \to C$ s.t. $r = w \cdot h_A$. Since $v \cdot w \cdot h_A = v \cdot r$ and $v \cdot w$ is strict, we have $v \cdot w = \mathrm{id}_{QA}$. On the other hand, we have $u \cdot w \cdot h_A = u \cdot r = \mathrm{id}_A = e_A \cdot h_A$, where the last equality comes from the triangle identities. It follows that $u \cdot w = e_A$.

We get then the invertible 2-cell $\rho_A = \lambda \cdot w\colon h_A \cdot e_A = h_A \cdot u \cdot w \Rightarrow\Rightarrow v \cdot w = \mathrm{id}_{QA}$, which satisfies $\rho_A \cdot h_A = \lambda \cdot r = 1_{h_A}$ by construction of r. This gives one of the triangle identities, while the other one follows formally from this since all 2-cells ??? are invertible as shown in the next lemma. □

Lemma 0.4.7. If $f\colon A \to B$, $u\colon B \to A$, $\eta\colon \mathrm{id}_A \Rightarrow u \cdot f$, $\epsilon\colon f \cdot u \Rightarrow \mathrm{id}_B$ are 1-cells and 2-cells s.t. $\epsilon f \cdot f\eta = 1_f$ and η, ϵ are invertible, then $u\epsilon \cdot \eta u = 1_u$ holds, so (f, u, η, ϵ) is an adjoint equivalence.

Proof. Since $u\cdot \cong \mathrm{id}$, f is faithful, so it suffices to checl that

$$\begin{tikzcd}
& \bullet \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \bullet \arrow[r, "f"] & \bullet \\
\bullet \arrow[ur, "u"] \arrow[rr, equal] & {} & \bullet \arrow[ur, "u"'] \arrow[u, phantom, "\Downarrow\eta"] & &
\end{tikzcd} \quad = \quad 1_{f\cdot u}$$

holds, which by invertibility of ϵ it is equivalent to

$$\begin{tikzcd}
& \bullet \arrow[rr, equal] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & & \bullet \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
\bullet \arrow[ur, "u"] \arrow[rr, equal] & {} & \bullet \arrow[ur, "u"'] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & {} & \bullet
\end{tikzcd} \quad = \quad \begin{tikzcd}
& \bullet \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
\bullet \arrow[ur, "u"] \arrow[rr, equal] & {} & \bullet
\end{tikzcd},$$

which follows from the fact that

$$\begin{tikzcd}
\bullet \arrow[rr, equal] \arrow[dr, "f"'] & {} & \bullet \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow\epsilon"] & \\
& \bullet \arrow[ur, "u"] \arrow[rr, equal] \arrow[u, phantom, "\Downarrow\eta"] & {} & \bullet
\end{tikzcd} \quad = \quad 1_f$$

□

This shows that the strict T-morphism $e_A\colon QA \to A$ is always an equivalence in $T\text{-}\mathsf{Alg}_P$; in fact, it is a surjective equivalence and it has an inverse equivalence which is a section.

Definition 0.4.8. An algebra A in $T\text{-}\mathsf{Alg}_S$ is flexible if $e_A\colon QA \to A$ has a section in $T\text{-}\mathsf{Alg}_S$.

Proposition 0.4.9. Assume that Q exists. The following are equivalent:

1. the T-algebra A is flexible;
2. the counit $e_A\colon QA \to A$ is a surjective equivalence in $T\text{-}\mathsf{Alg}_S$;
3. the T-algebra A is a retract of some QB in $T\text{-}\mathsf{Alg}_S$.

Proof. Missing □

The next theorem gives a first class of examples of flexible algebras.

Theorem 0.4.10. Suppose that Q exists and pseudolimits of arrows exist in $\mathcal{K}$. Then all free T-algebras are flexible.

Proof. We have that the free algebra 2-functor T is left adjoint to $U_S = U_P \cdot J$, with unit $\eta_A \colon A \to TA$. Consider the 2-natural transformation

$$\mathrm{id} \xRightarrow{\eta} U_P JT \xRightarrow{U_P hJT} U_P JQJT$$

with mate $k \colon T \Rightarrow QJT$. We claim that $eT \cdot k = \mathrm{id}_T$, so each e_{TA} is a retraction.

By definition, k is the unique 2-natural transformation s.t. $U_P Jk \cdot \eta = U_P hJT \cdot \eta$ holds, thus $U_P JeT \cdot U_P Jk \cdot \eta = U_P JeT \cdot U_P hJT \cdot \eta = \eta$ by the triangle identities, so by adjunction we get $eT \cdot k = \mathrm{id}$, as claimed.

It follows that e_{TA} is split by k_A and h is a 2-natural transformation between two 2-functors with target $T\text{-}\mathsf{Alg}_S$, so k_A is indeed a strict T-morphism. $\square$

Thus we have the examples TA and QA of flexible algebras. The following proposition, combined with the examples of free algebras, shows that free T-algebras in $T\text{-}\mathsf{Alg}_s$ are "essentially free" in $T\text{-}\mathsf{Alg}_p$: every $TA \rightsquigarrow B$ is isomorphic to a strict one, hence corresponds to $A \to B$ in $\mathcal{K}$.

Proposition 0.4.11. Assume Q exists plus pseudolimits of arrows in $\mathcal{K}$. If A is a flexible algebra, then the full and faithful inclusion $T\text{-}\mathsf{Alg}_s(A,B) \to T\text{-}\mathsf{Alg}_p(A,B)$ is essentially surjective for all B, hence an equivalence of categories. In other words, any $A \rightsquigarrow B$ is isomorphic to a strict T-morphism $A \to B$.

Proof. For algebras of the form QA, the commutative triangle

$$\begin{tikzcd} T\text{-}\mathsf{Alg}_s(QA,B) \arrow[r, "J_{QA,B}"] \arrow[dr, "\cong"'] & T\text{-}\mathsf{Alg}_p(JQA,JB) \arrow[d, "n_A^*"] \\ & T\text{-}\mathsf{Alg}_p(A,JB) \end{tikzcd}$$

coming from the 2-adjunction $Q \dashv J$, combined with the fact that n_A is an equivalence in $T\text{-}\mathsf{Alg}_p$, shows that $J_{QA,B}$ is an equivalence. For general flexible A, we know that e_A is an equivalence in $T\text{-}\mathsf{Alg}_s$, so the commutative square

$$\begin{tikzcd} T\text{-}\mathsf{Alg}_s(A,B) \arrow[r, "J_{A,B}"] \arrow[d, "e_A^*"', "\cong"] & T\text{-}\mathsf{Alg}_p(JA,JB) \arrow[d, "\cong"', "Je_A^*"] \\ T\text{-}\mathsf{Alg}_s(QA,B) \arrow[r, "\cong", "J_{QA,B}"'] & T\text{-}\mathsf{Alg}_p(JQA,JB) \end{tikzcd}$$

shows that $J_{A,B}$ is an equivalence of categories for all $B \in T\text{-}\mathsf{Alg}_s$. $\square$

Remark 0.4.12. Let T be the 2-monad on $\mathcal{V}$**-Cat** for e.g. finite conical limits. The above shows that TA is the free cocompletion of A under finite colimits: let $P_f A$ be the closure of the representables in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ under finite conical colimits. Then the induced

$$\begin{tikzcd} A \arrow[r, "y"] \arrow[dr, "\eta_A"'] & P_f A \arrow[d, dashed] \\ & TA \end{tikzcd}$$

is an equivalence. So, up to equivalence, η_A is the Yoneda embedding. In particular, η_A is fully faithful.

Definition 0.4.13. A *biequivalence* between bicategories (or 2-categories) consists of pseudofunctors $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{A}$ and pseudonatural equivalences $\mathrm{id}_{\mathcal{A}} \simeq GF$, $FG \simeq \mathrm{id}_{\mathcal{B}}$.

Remark 0.4.14. With the axiom of choice, $F\colon \mathcal{A} \to \mathcal{B}$ is a biequivalence if and only if each $F_{A,A'}\colon \mathcal{A}(A,A') \to \mathcal{B}(FA,FA')$ is an equivalence of categories and F is essentially surjective up to equivalence: for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ and an equivalence $FA \to B$ in $\mathcal{B}$.

We write $T\text{-}\mathsf{Flex}$ for the full sub-2-category of $T\text{-}\mathsf{Alg}_s$ consisting of the flexible algebras.

The above proposition, combined with the remark, shows that $J\colon T\text{-}\mathsf{Flex} \to T\text{-}\mathsf{Alg}_p$ is a biequivalence: every n_A is an equivalence in $T\text{-}\mathsf{Alg}_p$, so every object in the target is equivalent to one in the image, and the proposition shows $J_{A,A'}$ is an equivalence for A, A' flexible. In fact, much more is true in this case: the left 2-adjoint $Q\colon T\text{-}\mathsf{Alg}_p \to T\text{-}\mathsf{Alg}_s$ factors through $T\text{-}\mathsf{Flex}$ and n_A is an equivalence in $T\text{-}\mathsf{Alg}_p$, e_A is an equivalence in $T\text{-}\mathsf{Flex}$, so we have a biequivalence $F = J$, $G = Q$ s.t. $Q \dashv J$ (left 2-adjoint), n and e are 2-natural, F, G are 2-functors. The only "weakness" is that n_A, e_A are merely equivalences, not isomorphisms. If we are interested in properties of $T\text{-}\mathsf{Alg}_p$ that are invariant under biequivalences, it suffices to show that the corresponding property holds for $T\text{-}\mathsf{Flex}$. If we want to study cocompleteness properties of $T\text{-}\mathsf{Alg}_p$, it makes sense to first understand what kinds of colimits $T\text{-}\mathsf{Flex}$ has.

Recall that an *idempotent* $e\colon A \to A$ in a 2-category $\mathcal{K}$ is a 1-cell such that $e^2 = e$ (equality, not isomorphism). The *splitting* of this idempotent is the conical colimit of $(A, e)\colon \mathbf{Idem} \to \mathcal{K}$, where $\mathbf{Idem} = \{e\colon * \to *\}$ is the free category on one object with one idempotent.

Theorem 0.4.15. Let $\mathcal{K}$ be complete and cocomplete and let $T\colon \mathcal{K} \to \mathcal{K}$ be an accessible 2-monad. Then $T\text{-}\mathsf{Flex}$ is closed in $T\text{-}\mathsf{Alg}_s$ under splittings of idempotents, coproducts, coinserters, and coequifiers.

Proof. Splittings of idempotents are retracts, and $T\text{-}\mathsf{Flex}$ is clearly closed under retracts. The assumptions on $\mathcal{K}$ and T imply that $T\text{-}\mathsf{Alg}_s$ is cocomplete. Now let $j_n\colon A_n \to A$ be coproduct inclusions in $T\text{-}\mathsf{Alg}_s$ such that each A_n is flexible. Thus there exist 1-cells $h_n\colon A_n \to QA_n$ in $T\text{-}\mathsf{Alg}_s$ such that $e_{A_n} \cdot h_n = \mathrm{id}_{A_n}$. Let $h\colon A \to QA$ be the unique 1-cell such that $hj_n = Qj_n \cdot h_n$ for every n. Since $e\colon Q \Rightarrow \mathrm{id}$ is 2-natural, we have

$$e_A h j_n = e_A Q j_n h_n \overset{\text{2-nat}}{=} j_n \cdot e_{A_n} \cdot h_n = j_n$$

for all n, so $e_A h = \mathrm{id}_A$ therefore A is flexible. Now let $f, g\colon A \to B$ be 1-cells in $T\text{-}\mathsf{Alg}_s$ such that B is flexible (this turns out to suffice). Let $t\colon B \to C$ be the coinserter in $T\text{-}\mathsf{Alg}_s$, with 2-cell $\lambda\colon tf \Rightarrow tg$. We have to construct a section of $e_C\colon QC \to C$. Choose a section $k\colon B \to QB$ of $e_B\colon QB \to B$ in $T\text{-}\mathsf{Alg}_s$. Since $\mathcal{K}$ has pseudolimits of arrows, we have a 2-cell $\rho_B\colon n_B e_B \cong \mathrm{id}_{QB}$ with $e_B \rho_B = 1_{e_B}$. Thus $\sigma := \rho_B \cdot k\colon n_B \cong k$ is an isomorphism with $e_B \sigma = 1_{e_B k} = 1_{\mathrm{id}_B}$.

The 2-naturality of $n\colon \mathrm{id} \rightsquigarrow TQ$ gives $n_C t = Qt\colon n_B$, so we get a T-transformation

$$\tau\colon Qtk \cdot f \xRightarrow{Qt\sigma^{-1}f} Qtn_B f = n_C tf \xRightarrow{n_C \lambda} n_C tg = Qtn_B g \xRightarrow{Qt\sigma g} Qtk \cdot g$$

in $T\text{-}\mathsf{Alg}_s$ (since both the domain and codomain are strict T-morphisms). By the universal property of (C, t, λ), we get a unique 1-cell $h\colon C \to QC$ such that $ht = Qt \cdot k$ and $h\lambda = \tau$. Again using 2-naturality of e, we have $e_t ht = e_C Qtk = te_B k = t$ and $e_C h \lambda = e_C \tau$. We claim that $e_C \tau = \lambda$: this follows from $e_B \sigma = 1_{\mathrm{id}_B}$ and $e_C n_C = \mathrm{id}_C$. Thus the universal property tells us $e_C h = \mathrm{id}_C$, which shows that h is the desired section.

The coequifier is a bit easier: assume that we have two 2-cells $\alpha, \beta\colon f \Rightarrow g$ in $T\text{-}\mathsf{Alg}_s$ and again only that B is flexible. Let now $t\colon B \to C$ be the coequifier of α and β. Let $k\colon B \to QB$ and

$\sigma n_B \cong k$ be as above. Since $n_C t = Qtn_B$, we have $Qtn_B\alpha = Qtn_B\beta$ (by definition, $t\alpha = t\beta$), so $Qtk\alpha = Qth\beta$ because $n_B \cong k$. Thus there exists a unique $h\colon C \to QC$ such that $ht = Qt \cdot k$. Since

$$e_C ht = e_C Qt \cdot k \overset{\text{2-nat}}{=} te_B k = t,$$

we have $e_C h = \mathrm{id}_C$, so C is indeed flexible. □

Example 0.4.16. Let **Lex** be the 2-category of finitely complete categories and finite limits preserving functors (and all natural transformations). Here we assume that a fixed choice of limit has been made, so that this is $T\text{-}\mathsf{Alg}_p$ for some finitary 2-monad on **Cat**. We write $\mathbf{Lex}_s$ for the 2-category with the same 0-cells and 2-cells and with 1-cells the functors which strictly preserve the chosen limits; that is $T\text{-}\mathsf{Alg}_s$. We can start with the free category in $\mathbf{Lex}_s$ on one object G: this is flexible since all free algebras are. We can use a coinserter to get an object G and morphisms $e\colon * \to G$, $m\colon G \times G \to G$, and $i\colon G \to G$ (more precisely, we get a flexible algebra $\mathcal{B}$ such that $\mathbf{Lex}_s(\mathcal{B}, \mathcal{C})$ is isomorphic to the category of quadruples (G, e, m, i)). Finally, we use a coequifier to impose the laws of a group object: $m \cdot m \times G = m \cdot G \times m$, etc. Thus there is a flexible algebra G_p in $\mathbf{Lex}_s$ such that $\mathbf{Lex}_s(G_p, \mathcal{C})$ is isomorphic to the category of group objects in $\mathcal{C}$. Since every pseudomorphism $G_p \to \mathcal{C}$ is isomorphic to a strict one, we conclude that $\mathbf{Lex}(G_p, \mathcal{C})$ is equivalent to the category of group objects: G_p is the universal finitely complete category with a group object in it.

More details for the construction: the forgetful 2-functor $U_s\colon \mathbf{Lex}_s \to \mathbf{Cat}$ is represented by $T*$, the free T-algebra on one object. We start with the diagram

$$\begin{tikzcd} U_s \arrow[r, bend left, "{(*,-\times-,\mathrm{id})}"] \arrow[r, bend right, "{(\mathrm{id},\mathrm{id},\mathrm{id})}"'] & U_s \times U_s \times U_s \end{tikzcd}$$

in $[\mathbf{Lex}_s, \mathbf{Cat}]$. These are indeed 2-natural since each 1-cell in $\mathbf{Lex}_s$ strictly preserves $*$ and $-\times-$. It follows that they are induced by morphisms $T(*+*+*) \rightrightarrows T*$ in $\mathbf{Lex}_s$. If we form the coinserters of these two, then we obtain the category $\mathcal{B}$ above with $\mathbf{Lex}_s(\mathcal{B}, \mathcal{C}) \cong$ quadruples (G, e, m, i). Let $F = \mathbf{Lex}_s(\mathcal{B}, -)$. To impose the axioms, we consider

$$\begin{tikzcd} F \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow\ \Downarrow\beta"] & U_s \times U_s \times U_s \times U_s \end{tikzcd}$$

where the first component $\alpha_1, \beta_1\colon f_1 \Rightarrow g_1$ sends $\Gamma = (G, e, m, i)$ to $(G \times G) \times G = f_1(\Gamma)$, $g_1(\Gamma) = G$, α_1, β_1 are the ways of associating.

Example 0.4.17. Consider the 2-monad T on $[\mathcal{K}, \mathcal{K}]_\kappa$ with algebras 2-$\mathsf{Mnd}_\kappa(\mathcal{K})$, the 2-category of κ-accessible 2-monads. We have shown that the 2-monads for categories with chosen conical (co)limits and the 2-monads for (braided, symmetric) pseudomonoids are flexible algebras for T. On the other hand, the 2-monad for monoids is *not* flexible: we used a coequalizer in the construction, so this seems at least plausible.

Example 0.4.18. We now have an abstract reason why the free strict monoidal category on a monoid object exists. In the last course, we briefly indicated why this is given by Δ_+, the augmented simplex category with ordinal sum as tensor product. We can now give a rigorous proof of this. As in the case above for group objects, we can construct a flexible strict monoidal category $\mathcal{M}$ representing monoids. We claim that Δ_+ is isomorphic to $\mathcal{M}$; in particular, Δ_+ is flexible. Let (M, p, u) be the universal monoid in $\mathcal{M}$. We have to start with constructing a strict monoidal functor $\Delta_+ \to \mathcal{M}$ which sends $([0], \sigma_0, \delta_{-1})$ to (M, p, u). We set $F([n]) = M^{\otimes n+1}$, we send $\delta_i \colon [n] \to [n+1]$ to $M^{\otimes i} \otimes u \otimes M^{\otimes n-i+1} \colon M^{\otimes n+1} \to M^{\otimes n+2}$ and we send $\sigma_i \colon [n] \to [n-1]$ to $M^{\otimes i} \otimes p \otimes M^{\otimes n-i-1} \colon M^{\otimes n+1} \to M^{\otimes n}$ respectively. We have to check that the simplicial identities hold to show that F is a functor. This follows from the axioms for a monoid and the laws for a (strict) monoidal category (e.g. $\sigma_i\sigma_{i+1}$ needs the associativity axiom for p, while $\sigma_i\sigma_j$ $i < j-1$ uses only the axiom that $- \otimes - \colon \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is a functor). Next we need to check that the two diagrams

$$\begin{tikzcd} & * \arrow[dl, "{[-1]}"'] \arrow[dr, "I"] & \\ \Delta_+ \arrow[rr, "F"'] & & \mathcal{M} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Delta_+ \times \Delta_+ \arrow[r, "F \times F"] \arrow[d, "+"'] & \mathcal{M} \times \mathcal{M} \arrow[d, "\otimes"] \\ \Delta_+ \arrow[r, "F"'] & \mathcal{M} \end{tikzcd}$$

are commutative. The first is immediate from the definition. The second one clearly commutes on objects, so one only needs to check that it commutes on arrows of the form (σ_i, id), (δ_i, id), (id, σ_i), (id, δ_i). In all cases, we get $F(\sigma_i)$ resp. $F(\delta_i)$ tensored with some number of copies of M on the right resp. left; these numbers coincide for both composites in the diagram.

Since $([0], \sigma_0, \delta_{-1})$ is a monoid in Δ_+, there exists a $G \colon \mathcal{M} \to \Delta_+$ with $GM = [0]$, $Gp = \sigma_0$, $Gu = \delta_{-1}$. Thus $G(M^{\otimes n+1}) = [n]$. Thus GF is the identity on objects, and it is clearly full: each σ_i, δ_j arises from σ_0, δ_{-1} via ordinal sum on the right and left. Since the hom-sets are finite, it follows that GF is full and faithful and so GF is an isomorphism of categories: $GF_{[n],[m]} \colon \Delta_+([n],[m]) \to \Delta_+([n],[m])$. Moreover, FG sends (M, p, u) to (M, p, u) therefore by the universal property of $\mathcal{M}$ we have $FG \cong \mathrm{id}_{\mathcal{M}}$. Thus $\mathcal{M} \simeq \Delta_+$. This already shows that Δ_+ is flexible since it is a retract of $\mathcal{M}$. Moreover, the category of strong monoidal functors $\Delta_+ \to \mathcal{C}$ is *equivalent* to the category of monoids in $\mathcal{C}$.

We can extend this to non-strict monoidal categories: there exists a flexible monoidal category $\tilde{\mathcal{M}}$ which is free on a monoid. By MacLane's coherence theorem, there exists a strict monoidal category $\tilde{\mathcal{M}}'$ and a monoidal equivalence $\tilde{\mathcal{M}}' \to \tilde{\mathcal{M}}$. It follows that there exists a monoid in $\tilde{\mathcal{M}}'$ which is sent to a monoid isomorphic to the universal one. From the arguments above, we get a map $\Delta_+ \to \tilde{\mathcal{M}}'$ sending $([0], \sigma_0, \delta_{-1})$ to this monoid and thus an arrow $F \colon \Delta_+ \to \tilde{\mathcal{M}}'$ sending $([0], \sigma_0, \delta_{-1})$ to a monoid isomorphic to the universal one.

On the other hand, from the universal mapping property we get $G \colon \tilde{\mathcal{M}} \to \Delta_+$ sending the universal monoid to $([0], \sigma_0, \delta_{-1})$. The composite FG is then isomorphic to a strict monoidal functor and it follows from the universal mapping property that $FG \cong \mathrm{id}_{\tilde{\mathcal{M}}}$.

Note that Δ_+ has no non-identity isomorphisms, thus $GF \colon \Delta_+ \to \Delta_+$ is strict and it sends $([0], \sigma_0, \delta_{-1})$ to itself, hence $GF = \mathrm{id}_{\Delta_+}$.

Observe tat this does not imply that Δ_+ is flexible since F is not strict, however the argument shows that Δ_+ is equivalent to $\tilde{\mathcal{M}}$ in $T\text{-}\mathsf{Alg}_P$, hence $T\text{-}\mathsf{Alg}_P(\Delta_+, \mathcal{V}) \simeq \mathsf{Mon}(\mathcal{V})$.

0.5 Flexible Monads

Definition 0.5.1. A κ-accessible 2-monad on a locally κ-presentable 2-category is called flexible if it is a flexible algebra for the 2-monad for κ-accessible 2-monads in $\mathcal{K}$ with algebras $\mathsf{Mnd}_\kappa(\mathcal{K})$.

Proposition 0.5.2. If T is a felxbile 2-monad on $\mathcal{K}$, then any pseudo-T-algebra is isomorphic to a strict T-algebra.

Proof. Recall that $\mathsf{Mnd}_\kappa(\mathcal{K})$ is a coreflective subcategory in $\mathsf{Mnd}(\mathcal{K})$, so there exists a monad $\langle A, A\rangle_\kappa$ in $\mathsf{Mnd}_\kappa(\mathcal{K})$ s.t. we have a natural bijection between $T \to \langle A, A\rangle$ and $T \to \langle A, A\rangle_\kappa$ for all κ-accessible T and $A \in \mathcal{K}$. This bijection extends to pseudo-morphisms of monads (exercise), giving us a correspondance between $T \rightsquigarrow \langle A, A\rangle$ and $T \rightsquigarrow \langle A, A\rangle_\kappa$.

It follows that for any flexible T and any $\phi\colon T \rightsquigarrow \langle A, A\rangle)\kappa$ there exists a strict monad morphism $\tilde{\phi}\colon T \to \langle A, A\rangle_\kappa$ and an isomorpjism $\tau\colon \phi \Rightarrow \tilde{\phi}$.

The monad morphism ϕ corresponds to $a\colon TA \to A$, $\alpha\colon a \cdot Ta \Rightarrow a \cdot \mu_A$ and $\alpha_0\colon \mathrm{id}_A \Rightarrow a \cdot \eta_A$, i.e. a pseudo T-algebra, while $\tilde{\phi}$ corresponds to a strict T-algebra structure $\tilde{a}\colon TA \to A$. The morphism τ vorresponds to

$$\begin{tikzcd}
TA \arrow[r, "\tilde{a}"] \arrow[d, equal] \arrow[dr, phantom, "\Nearrow\tau"] & A \arrow[d, equal] \\
TA \arrow[r, "a"'] & A
\end{tikzcd},$$

which is a morphism of pseudo T-algebras since the equations

$$\begin{tikzcd}[column sep=small]
& T^2A \arrow[rr, "Ta"] \arrow[dl, equal] \arrow[dr, "\mu_A"] & & TA \arrow[dr, "a"] \arrow[d, phantom, "="] & \\
T^2A \arrow[dr, "\mu_A"'] \arrow[rr, phantom, "="] & & TA \arrow[rr, "a"] & & A \\
& TA \arrow[rr, "\tilde{a}"'] \arrow[ur, equal] & & A \arrow[ur, equal] \arrow[u, phantom, "="] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& T^2A \arrow[rr, "Ta"] \arrow[dl, equal] \arrow[d, phantom, "\Swarrow T\tau"] & & TA \arrow[dr, "a"] \arrow[dl, equal] \arrow[d, phantom, "\tau\Downarrow"] & \\
T^2A \arrow[dr, "\mu_A"'] \arrow[rr, "T\tilde{a}"] & & TA \arrow[dr, "\tilde{a}"] \arrow[d, phantom, "="] & & A \\
& TA \arrow[rr, "\tilde{a}"'] & & A \arrow[ur, equal] &
\end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=small]
& A \arrow[dl, equal] \arrow[dr, "\eta_A"] \arrow[rrr, equal, bend left, ""{name=U}] \arrow[rrr, phantom, bend left=10, "\alpha_0\Swarrow"] & & & A \\
A \arrow[dr, "\eta_A"'] \arrow[rr, phantom, "="] & & TA \arrow[urr, "a"] & & \\
& TA \arrow[rr, "\tilde{a}"'] \arrow[ur, equal] \arrow[urr, phantom, "\Swarrow\tau"] & & A \arrow[uur, equal] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& A \arrow[dl, equal] \arrow[drr, equal, bend left] \arrow[d, phantom, "="] & & \\
A \arrow[dr, "\eta_A"', bend right] \arrow[drr, equal, bend left=10] \arrow[dr, phantom, "="] & & & A \\
& TA \arrow[r, "\tilde{a}"'] & A \arrow[ur, equal] &
\end{tikzcd}$$

hold. □

Remark 0.5.3. For κ-accessible 2-monads on locally κ-presentable 2-categories we always have a bijection between $T \rightsquigarrow \langle A, A\rangle_\kappa$ and $QT \to \langle A, A\rangle_\kappa$, so pseudo T-algebras are strict QT-algebras. In this case we can work only with strict algebras without any loss of generality. Similarly, lax T-algebras correspond to strict $Q^L T$-algebras.

Example 0.5.4. The 2-monad for (braided/symmetric) monoidal pseudomonoids is flexible and the 2-monads for conical (co)limits of some shape are flexible, which is immediate from their presentations.

Example 0.5.5. The 2-monad for strict monoidal categories is not flexible. Indeed, each monoidal category gives rise to a pseudo T-algebra, where T is a strict monoid 2-monad on **Cat**, via $\Sigma\mathcal{M}^{\times n} \to \mathcal{M}$, with $\mathcal{M}^{\times n} \to \mathcal{M}$ given by $(M_1, \ldots, M_n) \mapsto ((M_1 \otimes M_2) \otimes \cdots)$ and α, α_0 for $(TM, a, \alpha, \alpha_0)$ given by the coherence theorem. This is in general not isomorphic via an identity-on-objects morphism to a strict monoidal category.

Proposition 0.5.6. If T is a flexible 2-monad on $\mathcal{K}$, then idempotents in $T\text{-}\mathsf{Alg}_P$ split, i.e. the equalizer of an idempotent e and the identity exists and is computed as in $\mathcal{K}$.

Proof. By a previous proposition, the inclusion $T\text{-}\mathsf{Alg}_P \to PsT\text{-}\mathsf{Alg}$ is an equivalence of **Cat**-enriched categories. It suffices to show that idempotents in $PsT\text{-}\mathsf{Alg}$ split, for then any strict T-algebra isomorphic to this splitting gives a splitting in $T\text{-}\mathsf{Alg}_P$.

For the splitting in $PsT\text{-}\mathsf{Alg}$, we can work with an arbitrary 2-monad.

Let then $(e, \bar{e})\colon (A, a) \to (A, a)$ be an idempotent in $T\text{-}\mathsf{Alg}_P$ and $s\colon B \to A$ be the the splitting of e in $\mathcal{K}$. From $e^2 = e$ we get $r\colon A \to B$ s.t. $s \cdot r = e$ and $s \cdot r \cdot s = e \cdot s = s$, thus $r \cdot s = \mathrm{id}_B$. We define $b\colon TB \to B$ as $TB \xrightarrow{Ts} TA \xrightarrow{a} A \xrightarrow{r} B$,

$$\beta_0 \quad = \quad \begin{tikzcd} B \arrow[r, "s"] \arrow[d, "\eta_B"'] \arrow[dr, phantom, "="] & A \arrow[d, "\eta_A"] \arrow[dr, equal] \arrow[dr, phantom, "="'] & & \\ TB \arrow[r, "Ts"'] & TA \arrow[r, "a"'] & A \arrow[r, "r"'] & B \end{tikzcd} \quad = \quad 1_{\mathrm{id}_B}$$

and

$$\beta \quad = \quad \begin{tikzcd} T^2B \arrow[r, "T^2s"] \arrow[d, "\mu_B"'] \arrow[dr, phantom, "="] & T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"] \arrow[dr, phantom, "="] & TA \arrow[r, "Tr"] \arrow[d, "a"] \arrow[dr, "Te"] \arrow[dr, phantom, "="', shift left=2] & TB \arrow[d, "Ts"] \\ TB \arrow[r, "Ts"'] & TA \arrow[r, "a"'] & A \arrow[dr, "e"'] \arrow[r, phantom, "\bar{e}\Downarrow"] & TA \arrow[d, "a"] \\ & & & A \arrow[d, "r"] \\ & & & B \end{tikzcd} .$$

We leave checking that (B, b, β, β_0) is an equalizer of $(e, \bar{e})$ and id_A in $PsT\text{-}\mathsf{Alg}$ as an exercis as an exercise. □

Remark 0.5.7. The above proposition, combined with our previous results on limits in $T\text{-}\mathsf{Alg}_P$, shows that $T\text{-}\mathsf{Alg}_P$ has products, inserters, equifiers and splittings of idempotents whenever T is flexible. Conversely, the 2-category $T\text{-}\mathsf{Flex}$ (which is biequivalent to $T\text{-}\mathsf{Alg}_P$) has the corresponding colimits for all accessible T. This naturally leads to the question:

which (co)limits can we build from these ingredients?

0.6 Flexible Colimits

Definition 0.6.1. A weight $W \in [\mathcal{A}, \mathbf{Cat}]$ is called flexible if it is a flexible algebra for the 2-monad arising from the adjunction $[\mathcal{A}, \mathbf{Cat}] \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} [\mathrm{Ob}(\mathcal{A}), \mathbf{Cat}]$. The colimits of lexible weights are called flexible (co)limits.

Definition 0.6.2. Let $D\colon \mathcal{A} \to \mathcal{K}$ be a small diagram in a 2-category $\mathcal{K}$, $W\colon \mathcal{A} \to \mathbf{Cat}$ any weight. A W-weighted lax limit (resp. pseudo limit) is a representing object $\{W, D\}_L$ (resp. $\{W, D\}_P$) for the 2-functor $C \mapsto [\mathcal{A}, \mathbf{Cat}]_L(W, \mathcal{K}(C, D))$ (resp. $[\mathcal{A}, \mathbf{Cat}]_P(W, \mathcal{K}(C, D))$. In other words, we have a 2-natural isomorphism $\mathcal{K}(C, \{W, D\}_L) \cong [\mathcal{A}, \mathbf{Cat}]_L(W, \mathcal{K}(C, D))$ (resp. $\mathcal{K}(C, \{W, D\}_P) \cong [\mathcal{A}, \mathbf{Cat}]_P(W, \mathcal{K}(C, D))$).

There is also an analogous notion of colax limits.

The notions of lax/pseudo/colax W-weighted colimits (e.g. $W \odot^P_{\mathcal{A}} D$) is defined dually in $\mathcal{K}^{\mathrm{op}}$.

Example 0.6.3. Let $\mathcal{A} = \{0 \to 1 \leftarrow 2\}$, $D = A \xrightarrow{f} B \xleftarrow{g} C$ in $\mathcal{K}$, $W = \Delta^1$. A pseudo-natural transformation $\Delta^1 \to \mathcal{K}(X, D-)$ amounts to three 1-cells and two invertible 2-cells

$$\begin{tikzcd} X \arrow[r, "k_2"] \arrow[d, "k_0"'] \arrow[dr, "k_1"] \arrow[dr, phantom, shift left=2, "\cong"] \arrow[dr, phantom, shift right=2, "\cong"] & C \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

and the pseudo pullback is the universal such diagram.

Note that this is similar to the iso-comma object, but it is not isomorphic to it: the two objects are just equivalent.

The following proposition shows that pseudo/lax limits do not give a new notion of limits: they still are weighted limits, but for a different weight.

Proposition 0.6.4. The lax (resp. pseudo) limit of a 2-functor $D\colon \mathcal{A} \to \mathcal{K}$ weighted by $W\colon \mathcal{A} \to$ **Cat** is given by the $Q^L W$-weighted colimit of D (resp. QW-weighted).

Proof. This follows fomr the defining isomorphism $[\mathcal{A}, \mathbf{Cat}]_L(W, F) \cong [\mathcal{A}, \mathbf{Cat}](Q^L, F)$ by specializing to the case $F = \mathcal{K}(C, D-)$. □

Proposition 0.6.5. If $\mathcal{K}$ has PIE-limits, then it has all lax and all pseudo limits.

Proof. We know that QW and $Q^L W$ can be built as iso- or lax codescent objects of free algebras and those can in turn be built from coinserters and coequifiers, so it remains to check that the W-weighted limit exists if W is free on a collection, which we prove in the next lemma. □

Lemma 0.6.6. Let $\mathcal{K}$ be a 2-category with PIE-limits. Then the W-weighted limit exists for all free algebras $W \in [\mathcal{A}, \mathbf{Cat}]$.

Proof. First note that the limit of $\mathcal{A}(a, -)$ always exists since $\{\mathcal{A}(a, -), D\} \cong Da$ by Yoneda. Moreover, $\mathcal{A}(a, -)$ is the free algebra on the collection $(\delta_a)_b = \emptyset$ if $b \neq a$, $= *$ if $b = a$ again by Yoneda. The class of weights W s.t. $\{W, D\}$ exists is closed under coproducts, coinserters and coequifiers since $\mathcal{K}$ has PIE-limits, which follows from $\{\operatorname{colim} W_i, D\} \cong \lim\{W_i, D\}$. Moreover, the left adjoint $T\colon [\operatorname{Ob}\mathcal{A}, \mathbf{Cat}] \to [\mathcal{A}, \mathbf{Cat}]$ preserves colimits, so the class of collections $(C_b)_{b\in\mathcal{A}}$ s.t. $T(C_b)_{b\in\mathcal{A}}$-weighted limits exist is also closed under coproducts, coinserters and coequifiers.

We have reduced the problem to showing that the closure of the δ_a under PIE-colimits is all of $[\operatorname{Ob}\mathcal{A}, \mathbf{Cat}]$. Using coproducts we can reduce to collections concentrated in a single "degree". This reduces the problem to the case $\operatorname{Ob}\mathcal{A} = *$, i.e. $[\operatorname{Ob}\mathcal{A}, \mathbf{Cat}] \cong \mathbf{Cat}$.

Every category can be written as a lax codescent object of its nerve, considered as a diagram of concrete categories. Clearly discrete categories are coproducts of the terminal category, so $*$ does indeed generate **Cat** under PIE-colimits. □

Corollary 0.6.7. If $\mathcal{K}$ has PIE-limits, $T\colon \mathcal{K} \to \mathcal{K}$ is a 2-monad, then $T\text{-}\mathsf{Alg}_P$ has all lax and pseudo limits.

Proof. We have shown that $T\text{-}\mathsf{Alg}_P$ has PIE-limits. □

Theorem 0.6.8. Let $\mathcal{K}$ be a 2-category with splittings of idempotents. Then $\mathcal{K}$ has all flexible limits. If $\mathcal{K}'$ is another 2-category with the same limits and $F\colon \mathcal{K} \to \mathcal{K}'$ is a 2-functor, which preserves PIE-limits (and splitting of idempotents, which is automatic), then F preserves all flexible limits.

Proof. From the lemma we know that $\{TC, D\}$ exists for all diagrams $D\colon \mathcal{A} \to \mathcal{K}$ and all collections $C = (C_b)_{b\in\mathcal{A}} \in [\mathrm{Ob}\,\mathcal{A}, \mathbf{Cat}]$, where T denotes the 2-monad for $[\mathcal{A}, \mathbf{Cat}]$. Since iso-descent objects can be built from coinserters and coequifiers, it follows that $\{QW, D\}$ exists for all weights W.

Finally, using splittings of idempotents, we find that $\{W, D\}$ exists for all flexible W: indeed, recalling that W is a retract of QW we can write it as a (co)splitting of an idempotent on QW.

Note that the proof so far actually shows that the flexible weights in $[\mathcal{A}, \mathbf{Cat}]$ are the closure of the representables under PIE colimits and (co)splittings of idempotents: the argument above shows one inclusion, while the other follows from the fact that $T\text{-}\mathsf{Flex} \subset T\text{-}\mathsf{Alg}_S$ is closed under PIE-colimits and splittings of idempotents.

To prove the second claim, we consider the class of all weights $W \in [\mathcal{A}, \mathbf{Cat}]$ (for a fixed $\mathcal{A}$) such that for all diagrams $D\colon \mathcal{A} \to \mathcal{K}$ the comparison morphism $F\{W, D\} \xrightarrow{\overline{F}} \{W, FD\}$ is an isomorphism. This contains the representables $\mathcal{A}(a, -)$ since both sides of the comparison map are then FDa (details left as an exercise) and the class is closed under PIE-colimits and splitting of idempotents because F preserves the corresponding limits by assumption. Since the closure of representables is the class of all flexible weights, it follows that F preserves flexible limits. $\square$

Remark 0.6.9. In the proof we have seen that flexible weights are precisely the closure of the representables under PIE-colimits and (co)splittings of idempotents. The second part of the theorem is purely a formal consequence of this.

Corollary 0.6.10. Let T be an accessible 2-monad on a complete and cocomplete 2-category $\mathcal{K}$. Then $T\text{-}\mathsf{Flex}$ is closed in $T\text{-}\mathsf{Alg}_S$ under all flexible colimits. In particular, flexible colimits of flexible monads are flexible and flexible colimits of flexible weights are flexible.

Proof. The theorem implies that $T\text{-}\mathsf{Flex}$ has all flexible colimits and the inclusion $T\text{-}\mathsf{Flex} \to T\text{-}\mathsf{Alg}_S$ preserves them. This simply means that a flexible colimit of lexible algebras, computed in $T\text{-}\mathsf{Alg}_S$, is again flexible. $\square$

Corollary 0.6.11. The weights for products, inserters, equifiers and splittings if idempotents are all flexible.

Proof. We know that representable weights are free, hence flexible. The Yoneda isomorphism $W \odot_{\mathcal{A}} Y \cong W$ shows that W is the W-weighted colimit of flexible weights. In particular, the weight for inserters can be written as coinserter of flexible weights, so it is flexible by the theorem that $T\text{-}\mathsf{Flex}$ is closed under coinserters. The other weights follow in the same way. $\square$

We can summarize our results about limits in $T\text{-}\mathsf{Alg}_P$ as follows:

Proposition 0.6.12. If $\mathcal{K}$ has PIE-limits, then so does $T\text{-}\mathsf{Alg}_P$ and they are preserved by $U_P\colon T\text{-}\mathsf{Alg}_P \to \mathcal{K}$. This class includes all pseudo and lax limits. If $\mathcal{K}$ is locally κ-presentable and T is flexible, then $T\text{-}\mathsf{Alg}_P$ has all flexible limits and they are preserved by U_P.

Proof. We proved the statements about pseudo/lax limits above. The second statement follows from the fact that $T\text{-}\mathsf{Alg}_P$ has splittings of idempotents if T is flexible. U_P preserves them, so the claim follows from the above theorem. $\square$

Remark 0.6.13. Even though $T\text{-}\mathsf{Flex}$ has all flexible colimits, the same is not true for the biequivalent 2-category $T\text{-}\mathsf{Alg}_P$.

Example 0.6.14. Consider the 2-monad T on **Cat** with $T\text{-}\mathsf{Alg}_P = \mathbf{Lex}$, with finitely complete categories and finite limit preserving (i.e. left exact) functors. If **Lex** had flexible limits, it would have a pseudo initial object, but any pseudo initial object is a strict initial object and **Lex** has no such thing: the two functors $c_0, c_1 \colon \mathcal{A} \to \{0 \cong 1\}$ are left exact and distinct for all finitely complete (in fact all non-empty) $\mathcal{A}$.

0.7 Weak limits and bilimits

Definition 0.7.1. Let $\mathcal{A}, \mathcal{B}$ be bicategories, $W \colon \mathcal{A} \to \mathbf{Cat}$ and $D \colon \mathcal{A} \to \mathcal{B}$ pseudofunctors. A *W-weighted bilimit* of D (also called *weak limit* or *bicategorical limit*) is an object $\{W, D\}_b$ with a pseudonatural equivalence

$$\mathcal{B}(B, \{W, D\}_b) \simeq \mathrm{Ps}[\mathcal{A}, \mathbf{Cat}](W, \mathcal{B}(B, D-))$$

of categories. The notion of bicolimit is defined dually in $\mathcal{B}^{\mathrm{op}}$.

Example 0.7.2. A bi-initial object is an object $I \in \mathcal{B}$ s.t. each $\mathcal{B}(I, B)$ is equivalent to the terminal category. we have $\mathrm{Ps}[\mathcal{A}, \mathcal{B}] = *$, so $\mathrm{Ps}[\mathcal{A}, \mathcal{B}](W, \mathcal{B}(D, -)) \simeq *$ for all B (NOT SURE ABOUT THIS, I THINK IT SHOULD BE: $\mathrm{Ps}[A, \mathbf{Cat}](W, \mathcal{B}(D-, B)) \simeq \mathcal{B}(I, B) \simeq *$ for all B)

Note that **Lex** does have a bi-initial object given by $*$, since any left exact functor $* \to \mathcal{C}$ sends $*$ to a terminal object and all terminal objects are uniquely isomorphic.

Proposition 0.7.3. If $\mathcal{A}, \mathcal{K}$ are 2-categories, $W \colon \mathcal{A} \to \mathbf{Cat}$, $D \colon \mathcal{A} \to \mathcal{K}$ is a 2-functor and the pseudolimit of D weighted by W exists, then $\{W, D\}_p$ is a bilimit.

Proof. Note that $[\mathcal{A}, \mathbf{Cat}]_p \subseteq \mathrm{Ps}[\mathcal{A}, \mathbf{Cat}]$ is a full sub-2-category, so

$$\mathrm{Ps}[\mathcal{A}, \mathbf{Cat}](W, \mathcal{K}(B, D-)) \simeq [\mathcal{A}, \mathbf{Cat}]_p(W, \mathcal{K}(B, D-))$$

under our assumptions. Clearly any 2-natural isomorphism is in particular a pseudonatural equivalence. □

From the bicategorical Yoneda lemma it follows that bilimits are unique up to essentially unique equivalence. Thus in the above situation any bilimit is equivalent to the pseudolimit.

Remark 0.7.4. In general $\{W, D\} \not\simeq \{W, D\}_b$. For example let $W = \Delta_1$ be the conical weight on the span category. The diagram

$$\begin{tikzcd} & * \arrow[d] \\ * \arrow[r] & \{0 \cong 1\} \end{tikzcd}$$

in **Cat** has $\{W, D\} =$, but the diagram

$$\begin{tikzcd} * \arrow[r, equal] \arrow[d, equal] \arrow[dr, "1"] \arrow[dr, phantom, shift right=3, "\cong"] & * \arrow[d, "1"] \arrow[dl, phantom, shift right=2, "="] \\ * \arrow[r, "0"'] & \{0 \cong 1\} \end{tikzcd}$$

defines a pseudocone, so $\{W, D\}_p \simeq \{W, D\}_b$ is *not* empty.

Proposition 0.7.5. Let $\mathcal{K}$ be a 2-category with flexible limits, $\mathcal{A}$ be a 2-category and $W\colon \mathcal{A} \to \mathbf{Cat}$, $D\colon \mathcal{A} \to \mathcal{K}$ be 2-functors. If W is flexible then $\{W, D\} \simeq \{W, D\}_b$. In other words, flexible limits are bilimits.

Proof. Since $e_W\colon QW \to W$ is a surjective equivalence in $[\mathcal{A}, \mathbf{Cat}]$, it induces an equivalence

$$\begin{aligned}\mathcal{K}(C, \{W, D\}) &\simeq [\mathcal{A}, \mathbf{Cat}](W, \mathcal{K}(C, D-))\\ &\overset{e_W^*}{\simeq} [\mathcal{A}, \mathbf{Cat}](QW, \mathcal{K}(C, D-))\\ &\simeq [\mathcal{A}, \mathbf{Cat}]_p(W, \mathcal{K}(C, D-))\\ &\simeq \mathcal{K}(C, \{W, D\}_p).\end{aligned}$$

Since pseudolimits are bilimits, the conclusion follows. □

Using this notion we can show that $T\text{-}\mathsf{Alg}_p$ is weakly cocomplete.

Theorem 0.7.6. Let $\mathcal{K}$ be complete and cocomplete and T an accessible 2-monad. Let $W\colon \mathcal{A} \to \mathbf{Cat}$, $D\colon \mathcal{A} \to T\text{-}\mathsf{Alg}_p$ be 2-functors, with $\mathcal{A}$ small 2-category. Then $W \odot^p_{\mathcal{A}} Q \circ D$ in $T\text{-}\mathsf{Flex}$ is a bicolimit $W \odot^b_{\mathcal{A}} D$ of D weighted by W in $T\text{-}\mathsf{Alg}_p$.

Proof. Note that $W \odot^p_{\mathcal{A}} Q \circ D$ is given by $Q(W) \odot_{\mathcal{A}} Q \circ D$ (where the first Q is the pseudomorphism classifier in $[\mathcal{A}, \mathbf{Cat}]$, while the second one is for $T\text{-}\mathsf{Alg}_s$). Since $Q(W)$ is flexible, so is $W \odot^p_{\mathcal{A}} Q \circ D$. From this we get isomorphisms and equivalences as follows

$$\begin{aligned}T\text{-}\mathsf{Alg}_p(W \odot^p_{\mathcal{A}} Q \circ D, \mathcal{A}) &\xleftarrow{\simeq} T\text{-}\mathsf{Alg}_s(W \odot^p_{\mathcal{A}} Q \circ D, \mathcal{A})\\ &\cong [\mathcal{A}, \mathbf{Cat}]_p(W, T\text{-}\mathsf{Alg}_s(Q \circ D-, \mathcal{A}))\\ &\cong [\mathcal{A}, \mathbf{Cat}]_p(W, T\text{-}\mathsf{Alg}_p(D-, \mathcal{A}))\end{aligned}$$

which are 2-natural in $\mathcal{A}$. The first one has a pseudonatural inverse, so $W \odot^p_{\mathcal{A}} Q \circ D$ is indeed a bicolimit of D weighted by W. □

Corollary 0.7.7. In the above situation, $T\text{-}\mathsf{Alg}_p$ has *all* small bilimits and bicolimits.

Proof. The theorem shows that $T\text{-}\mathsf{Alg}_p$ has bicategorical coproducts, bicoequalizers and bicopowers by small categories, since the diagrams for all these can be chosen to be strict 2-categories and 2-functors. Ross Street showed in the *Errata to "Filtrations in bicategories"* that these can be used to construct *all* small bicolimits. The claim about limits follows analogously (we have all pseudolimits!). □

We have the following result about preservation of bilimits.

Proposition 0.7.8. Any biequivalence preserves bilimits and bicolimits.

Proof. The usual proof "categorifies": given a biequivalence $F\colon \mathcal{K} \to \mathcal{L}$, $D\colon \mathcal{A} \to \mathcal{K}$ a diagram, $W\colon \mathcal{A} \to \mathbf{Cat}$ a weight, we have

$$\begin{aligned}\mathcal{L}(X, F\{W, D\}_b) &\simeq \mathcal{L}(FY, F\{W, D\}_b) && (F \text{ essentially surjective})\\ &\simeq \mathcal{K}(Y, \{W, D\}_b) && (F \text{ equivalence on Hom-categories})\\ &\simeq \mathrm{Ps}[\mathcal{A}, \mathbf{Cat}](W, \mathcal{K}(Y, D-))\\ &\simeq \mathrm{Ps}[\mathcal{A}, \mathbf{Cat}](W, \mathcal{L}(FY, FD-))\\ &\simeq \mathrm{Ps}[\mathcal{A}, \mathbf{Cat}](W, \mathcal{L}(X, FD-))\end{aligned}$$

so $F\{W, D\}_b \simeq \{W, FD\}_b$ by bicategorical Yoneda. □

Remark 0.7.9. More generally, left biadjoints preserve bicolimits and right biadjoints preserve bilimits.

Corollary 0.7.10. Each flexible colimit in T-Flex is a bicolimit in T-Alg_p.

Proof. We know that flexible colimits are bicolimits, so they are preserved by the biequivalence T-$\mathsf{Flex} \to T$-Alg_p. □

Example 0.7.11. The diagram

$$T^3A \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; T^2A \; \underset{Ta}{\overset{\mu_A}{\rightrightarrows}} \; TA \xrightarrow{\;a\;} A$$

exhibits A as bicategorical iso-codescent object of

$$\begin{aligned} \overline{\Delta_{\leq 2}} &\to T\text{-}\mathsf{Alg}_p \\ [n] &\mapsto T^{n+1}A. \end{aligned}$$

To see this note that QA is the (strict) iso-codescent object of the diagonal in T-Flex, so QA is also a bicategorical iso-codescent object in T-Alg_p. Composing with the equivalence $e_A \colon QA \to A$ in T-Alg_p, we obtain the above diagram. It is still a bicategorical iso-codescent object, since these are only defined up to equivalence.

To summarize, we have shown that $U_p \colon T\text{-}\mathsf{Alg}_p \to \mathcal{K}$ has many of the nice properties of $U \colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ of 1-monads:

(i) From the exercises we know that U_p is "conservative", i.e. it reflects equivalences in $\mathcal{K}$.

(ii) If $\mathcal{K}$ is complete and cocomplete and T is accessible, then T-Alg_p is bicategorically complete and cocomplete.

(iii) The diagram

$$T^3A \; \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \; T^2A \; \underset{Ta}{\overset{\mu_A}{\rightrightarrows}} \; TA \xrightarrow{\;a\;} A$$

shows that each algebra is canonically a bicolimit of free algebras.

(iv) The following lemma shows that $U_p \colon T\text{-}\mathsf{Alg}_p \to \mathcal{K}$ also has a left biadjoint.

Lemma 0.7.12. If $J \colon \mathcal{A} \to \mathcal{B}$ is a biequivalence with inverse Q and $FJ \simeq \mathcal{A}(A, -)$, then $F \simeq \mathcal{B}(JA, -)$. In particular, we have $T\text{-}\mathsf{Alg}_p(JTA, -) \simeq \mathcal{K}(A, U_p-)$.

Proof. The second claim follows from the first. In fact, taking $F \cong \mathcal{K}(A, U_p-)$ yields $FJ \cong T\text{-}\mathsf{Alg}_s(TA, -)$. For the first claim we have

$$B(JA, -) \simeq \mathcal{A}(QJA, Q-) \simeq \mathcal{A}(A, Q-) \simeq FJQ(-) \simeq F$$

since $QJ \simeq \mathrm{id}$ and $JQ \simeq \mathrm{id}$. □

Remark 0.7.13. One can say a little more about this left biadjoint: it is given by the 2-functor JT and the functors $T\text{-}\mathsf{Alg}_p(JTA, B) \to \mathcal{K}(A, U_pB)$ are surjective equivalences. This follows from Theorem 5.1 in Blackwell-Kelly-Power's "Two-dimensional monad theory".

We saw $U_p \colon T\text{-}\mathsf{Alg}_p \to \mathcal{K}$ behaves very similar to $U \colon T\text{-}\mathsf{Alg} \to \mathcal{C}$ for 1-monads on a 1-category $\mathcal{C}$, if we work up to equivalence everywhere. In practice, it is often convenient to work with the stricter structures such as T-Flex. For example, constructing the free finitely complete category with a group object on the free monoidal category with a monoid is harder to do if we only use bicolimits in T-Alg_p.

0.8 Coherence Theorems

A major theme of this course was the use of strict structures to study weak ones. A natural question we can ask is the following: *under what conditions on the* 2*-monad* T *is every pseudo* T*-algebra equivalent to a strict one?*

We have shown that this is true for flexible T, but this is not necessary: for example MacLane's coherence theorem implies that the 2-monad for strict monoidal categories also has this property.

In the case where $\mathcal{K}$ is locally presentable and T accessible, we can make this more precise.

Let $T' = QT$ be the pseudomorphism classifier of T in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})$. Then we have an equivalence $e_T \colon T' \to T$ in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})_p$ and an isomorphism of 2-categories $PsT\text{-}\mathsf{Alg} \cong T'\text{-}\mathsf{Alg}_p$. Moreover $e_T^* \colon T\text{-}\mathsf{Alg}_s \to T'\text{-}\mathsf{Alg}_s$ has a left 2-adjoint $(e_T)_*$. Since

$$T\text{-}\mathsf{Alg}_s \to T'\text{-}\mathsf{Alg}_s \xrightarrow{J} T'\text{-}\mathsf{Alg}_p$$

corresponds to the natural inclusion $T\text{-}\mathsf{Alg}_s \to PsT\text{-}\mathsf{Alg}$, this inclusion has a left 2-adjoint $(e_T)_*Q$.

Definition 0.8.1. We say that the full coherence theorem holds for T if the unit of the above adjunction is an equivalence in $PsT\text{-}\mathsf{Alg}$. In this case every pseudo T-algebra is canonically equivalent to a strict one.

Proving this coherence theorem is possible if we make some additional assumptions on T, for example that T preserves certain iso-codescent objects. Generalizing slightly, we can consider an arbitrary monad morphism $\varphi \colon S \to T$ (instead of $e_T \colon T' \to T$).

Proposition 0.8.2. Let $\mathcal{K}$ be complete and cocomplete, S, T accessible. If both S and T preserve iso-codescent objects of reflexive coherence data (keeping track of the degeneracies), then the unit of the 2-adjunction

$$\begin{tikzcd} T\text{-}\mathsf{Alg}_s \arrow[r, shift left, "\varphi^*"] & S\text{-}\mathsf{Alg}_s \arrow[l, shift left, "\varphi_*"] \arrow[r, shift left, "J"] & S\text{-}\mathsf{Alg}_p \arrow[l, shift left, "Q"] \end{tikzcd}$$

is an equivalence if and only if each $\varphi_K \colon SK \to TK$ is an equivalence in $\mathcal{K}$.

Proof. The unit is given by the composition $A \overset{\eta_A}{\rightsquigarrow} JQA \xrightarrow{J\eta_{QA}} \varphi^*\varphi_*QA$ so this is an equivalence for every A if and only if each η_{QA} is an equivalence. Since both S, T preserve iso-codescent objects of reflexive codescent data, so do U_s and φ^*. Since these are flexible colimits and QA *is* such an iso-codescent object of a reflexive codescent datum where each algebra is free, it suffices to check that $\eta_{SK} \colon SK \to \varphi^*\varphi_*SK$ is an equivalence in $S\text{-}\mathsf{Alg}_p$ for all $K \in \mathcal{K}$. This unit is up to isomorphism given by $SK \xrightarrow{\varphi_K} \varphi^*TK$, which is an equivalence in $\mathcal{K}$ by assumption, hence an equivalence in $S\text{-}\mathsf{Alg}_p$. The converse follows, since SK is a retract of QSK in $S\text{-}\mathsf{Alg}_s$, so unit equiv. $\Rightarrow \eta_{QSK}$ equiv. $\Rightarrow \eta_{SK}$ equiv. $\Rightarrow \varphi_K$ equiv. □

Proposition 0.8.3. Let $\mathcal{K}$ be a locally κ-presentable 2-category and $F \colon \mathcal{K} \to \mathcal{K}$ a κ-accessible 2-functor. If F preserves W-weighted colimits for some weight W, then so does the free κ-accessible 2-monad $T(F) \colon \mathcal{K} \to \mathcal{K}$ on F.

Proof. The underlying 2-functor of $T(F)$ is given as colimit of the chains $X_\beta, \beta < \kappa$ defined by $X_0 = \mathrm{id}_\mathcal{K}$, $X_{\beta+1} = \mathrm{id}_\mathcal{K} + F \circ X_\beta$, $X_\alpha = \mathrm{colim}_{\beta<\alpha} X_\beta$ for limit ordinals (Kelly). By transfinite induction we see that each X_β preserves W-weighted colimits. □

Definition 0.8.4. A weight W is called sifted, if $W \odot_{\mathcal{A}} -\colon [\mathcal{A}, \mathbf{Cat}] \to \mathbf{Cat}$ preserves finite products.

Using this, one can show that the left Kan extension of W along the diagonal $\Delta\colon \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ is $W \times W$ (pointwise product). This implies the usual diagonal lemma for sifted colimits and therefore that $X \hookrightarrow F(X, X)$ preserves W-colimits if both $F(-, X)$ and $F(X, -)$ preserves W-colimits for each fixed X.

Using this we can prove the following

Lemma 0.8.5. If $D\colon \mathcal{A} \to 2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})$ is a diagram s.t. each D_a preserves W-weighted colimits and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat}$ is sifted, then $W \odot_{\mathcal{A}} D$ in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})$ also preserves W-weighted colimits. More precisely this colimit is preserved by $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K}) \to [\mathcal{K}, \mathcal{K}]_\kappa$. This can be done, since $W \odot_{\mathcal{A}} D \circ W \odot_{\mathcal{A}} D$ is the W-weighted colimit of the diagonal $a \mapsto D_a \circ D_a$ (by our assumption that W is sifted). To check associativity one uses the fact that this also holds true for the triple composite.

Remark 0.8.6. A more conceptual argument is possible if the $\{W\}$-cocontinuous κ-accessible endo-2-functors are coreflective in $[\mathcal{K}, \mathcal{K}]_\kappa$, for when we get a monoidal 2-adjunction, which lifts to 2-categories of monoids: thus the inclusion

$$2\text{-}\mathsf{Mnd}_{\{W\}\mathrm{cocts}}(\mathcal{K}) \to 2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})$$

has a right adjoint, so all colimits are preserved by this inclusion. This holds in many cases of interest and it always holds, if we assume Vopěnka's principle.

Some examples of sifted colimits are lax/iso codescent objects of reflexive codescent data and reflexive inverters

$$\begin{tikzcd} \bullet \arrow[r, "r"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \quad = \quad 1_{\mathrm{id}}$$

Corollary 0.8.7. If the κ-accessible 2-monad T preserves reflective codescent objects, then so does $QT = T'$.

Proof. Combine the proposition and the lemma: QT is a reflective iso-codescent object of a diagram of iterated free monads on T. ☐

Theorem 0.8.8. The full coherence theorem holds for accessible 2-monads on a locally presentable $\mathcal{K}$, which preserve iso-codescent objects of reflexive codescent data.

Proof. $T' = QT$ and T preserves reflexive iso-codescent objects, so we only need to show that $(e_T)_K\colon T'K \to TK$ is an equivalence in $\mathcal{K}$. This follows from the fact that e_T is a surjective equivalence in $2\text{-}\mathsf{Mnd}_\kappa(\mathcal{K})_p$, so in particular an equivalence in $[\mathcal{K}, \mathcal{K}]_\kappa$. ☐

Remark 0.8.9. A more direct argument is given by Lack in "Codescent objects and coherence" without using locally presentable categories, but a weaker form of codescent diagrams.

Applying these arguments to the 2-monads for strict monoids and pseudomonoids we can reduce the coherence theorem for such to the case of free (pseudo-)monoids.

Let $\mathcal{K}$ be a locally κ-presentable 2-category with a monoidal structure s,t, $- \otimes -$ preserves κ-filtered colimits in each variable. Let T be the 2-monad for strict monoids, S the 2-monad for pseudomonoids. Assume that $- \otimes -$ preserves iso-codescent objects of reflexive codescent diagrams (or maybe even all sifted colimits). Finally, assume that the F-cocontinuous endo-2-functors in $[\mathcal{K}, \mathcal{K}]_\kappa$ are coreflective for $F = \{\text{refl. iso-codescent sifted}\}$. It follows that both

S and T preserve iso-codescent objects of reflective codescent diagrams. This is for example the case for $F = \{\text{sifted}\}$, $\mathcal{K} = \mathbf{Cat}/\mathbf{Cat}^{X \times X}$. Showing that each pseudomonoid is equivalent to a strict one now boils down to checking that each $SK \to TK$ is an equivalence in $\mathcal{K}$. In the case of $\mathbf{Cat}/\mathbf{Cat}^{X \times X}$ we can reduce further to levelwise finite discrete categories. This follows from the fact that each category is the lax codescent object of its nerve, considered as a diagram of discrete categories. So we have reduced the coherence theorem to checking that the natural strong monoidal functor from the free monoidal category on $\{1, \ldots, n\}$ (which is simply the discrete category on the free monoid on $\{1, \ldots, n\}$) is an equivalence. As in the case of the universal property of Δ_+, the hard part is constructing the strong monoidal functor $T\{1, \ldots, n\} \to S\{1, \ldots, n\}$. Writing A_i for the object corresponding to i, we can send a word in the A_i to the corresponding word with "leftmost" bracketing (empty word $\mapsto$ unit). Using associators and unitors, it is possible to write down the data of a monoidal functor on this. Checking that this is monoidal is the essence of MacLane's coherence theorem.

Question: Is every monoidal $\mathcal{V}$-category ($\mathcal{V}$ lfp cosmos) equivalent to a strict monoidal $\mathcal{V}$-category?