HOPF INVARIANT ONE

HOPF INVARIANT

- $\pi_k(S^n) = 0 \quad \forall k < n$ (cellular approximation)
- $\pi_n(S^n) \cong \mathbb{Z}$ (Freudenthal + degree)

$\pi_k(S^n) \quad k > n$?

$\pi_3(S^2)$ is not trivial

You can construct

$$\mathbb{C}P^{n-1} = S^{2n-1} / \sim$$

$x, y \in S^{2n-1}$
$x \sim y$
$\iff \exists \lambda \in \mathbb{C}$
s.t. $x = \lambda y$

$$x \sim y \implies x = \lambda y$$

$$1 = \|x\| = \|\lambda y\| = |\lambda| \underbrace{\|y\|}_{=1} = |\lambda|$$

$$S^{2n-1} \longrightarrow \mathbb{C}P^{n-1}$$

$\boxed{n = 2}$

$$p : S^3 \longrightarrow \mathbb{C}P^1 \cong S^2 = \mathbb{C} \cup \{\infty\}$$

$$(z_0, z_1) \longmapsto \frac{z_0}{z_1}$$

p is called Hopf fibration (or Hopf bundle)

$\pi_3(S^2)$ IS NOT TRIVIAL

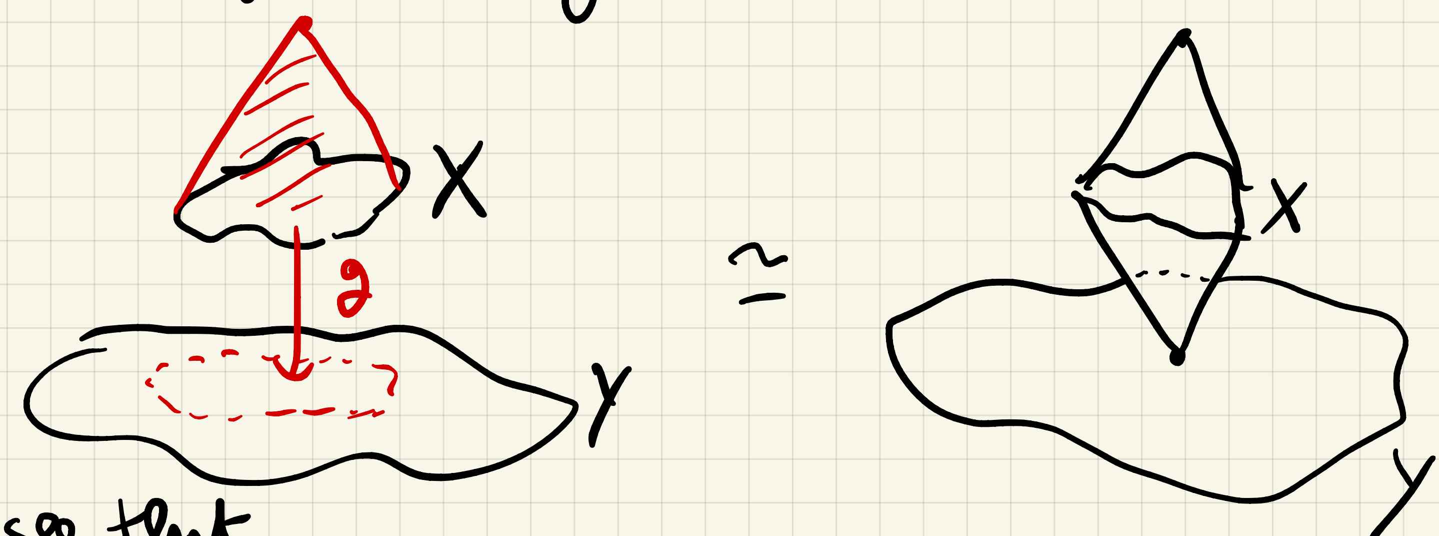
if $\pi_3(S^2)$ was trivial

EVERY $f: S^3 \to S^2$ would be null-homotopic (in particular, the Hopf fibration p)

<u>Rmk</u>

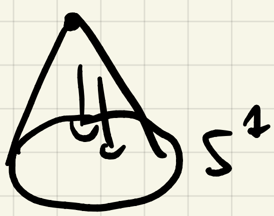
if $g: X \to Y$ is null-homotopic

$$\mathrm{Cone}(g) = CX \cup_g Y \simeq SX \vee Y$$



We see that

$$\mathrm{Cone}(p) = CS^3 \cup_p S^2 \simeq D^4 \cup_p S^2 \simeq \mathbb{CP}^1 \cup_p e^4$$
$$\simeq \mathbb{CP}^2$$



$$SS^3 \vee S^2 \simeq S^4 \vee S^2$$

but $H^*(\mathbb{CP}^2) \neq H^*(S^4 \vee S^2)$ AS RINGS

HOPF INVARIANT

$p: S^3 \to S^2$

$f: S^{2n-1} \to S^n$

Since $S^{2n-1} \cong \partial D^{2n}$, we can attach a $2n$-cell to S^n via f $K := S^n \cup_f e^{2n}$

$$H^k(K, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, n, 2n \\ 0 & \text{else} \end{cases}$$

$\langle \sigma \rangle = H^n(K, \mathbb{Z})$

$\langle \tau \rangle = H^{2n}(K, \mathbb{Z})$

$$H^n(K, \mathbb{Z}) \longrightarrow \overbrace{H^{2n}(K, \mathbb{Z})}^{\langle \tau \rangle}$$
$$\sigma \longmapsto \sigma^2$$

$$\sigma^2 = H(f)\tau$$

Def $H(f)$ is called the Hopf invariant of f

Rmk If n is odd, then $H(f) = 0$

In fact $xy = (-1)^{\deg x \deg y} yx$

$$\sigma^2 = (-1)^{n^2}\sigma^2 = -1\cdot\sigma^2 = -\sigma^2$$

$$\Rightarrow \sigma^2 = 0 \Rightarrow H(f) = 0$$

<u>Rmk</u> The homotopy type of K only depends on the homotopy class of f $\Rightarrow$ $H(f)$ depends only on the homotopy class of f

therefore we have a $\boxed{\text{function}}$

$$H : \pi_{2n-1}(S^n) \longrightarrow \mathbb{Z}$$

<u>Theorem</u> There is no map of Hopf invariant one for $n \neq 2, 4, 8$

<u>Theorem</u> If $f : S^{2n-1} \longrightarrow S^n$ is s.t.

$H(f) = 1 \Rightarrow n$ is a power of 2

prop If n is even, there exist a map $f: S^{2n-1} \to S^n$ s.t. $H(f) = 2$

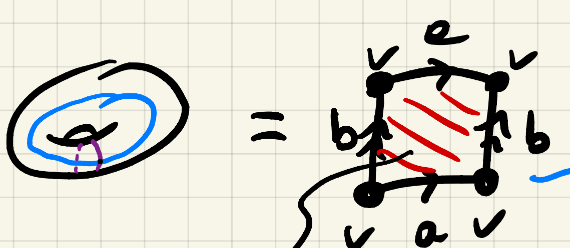
proof

$$\begin{tikzcd} S^{2n-1} \arrow[r, "f"] \arrow[dr, "g"'] & S^n \\ & S^n \vee S^n \arrow[u, "\nabla"'] \end{tikzcd}$$

$\nabla = (id, id)$

g is the map used to construct $S^n \times S^n = (S^n \vee S^n) \cup_g e^{2n}$

$n=1$ $S^1 \times S^1$



$\rightsquigarrow e^0$

e^1_1, e^1_2

e^2

$K = S^n \cup_f e^{2n}$ we have to show that $\boxed{\sigma^2 = 2\tau}$

claim $H(f)=2$

$i: S^n \hookrightarrow K$

$S^n \times S^n \longrightarrow K$

$i \nabla g: S^{2n-1} \longrightarrow K$ null-homotopic

$(S^n \vee S^n) \cup_g e^{2n} \xrightarrow{\overline{\nabla}} K = S^n \cup_f e^{2n}$

$\overline{\nabla}$ agrees with ∇ on $S^n \vee S^n$

$\overline{\nabla}$ is a relative homeo on (e^{2n}, S^{2n-1})

$\Rightarrow \overline{\nabla}^*$ is an iso in dimension $2n$

$$\boxed{\overline{\nabla}^* \sigma^2 =} \ \overline{\nabla}^*(H(f)\tau) = H(f)\,\underbrace{\overline{\nabla}^*(\tau)}_{\rho \text{ is the generator of } H^{2n}(S^n \times S^n)} = H(f)\rho$$

σ_1, σ_2 generators of $H^n(S^n)$

$$\boxed{\overline{\nabla}^*(\sigma^2) =} \ (\overline{\nabla}^*\sigma)^2 = (\nabla^*\sigma)^2 = (\sigma_1 + \sigma_2)^2 =$$

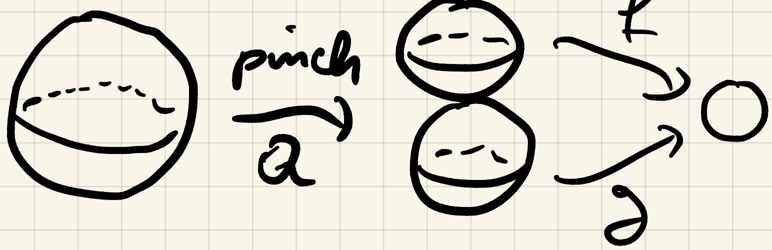
$$= \underset{\substack{\shortparallel \\ 0}}{\sigma_1^2} + \underset{\substack{\shortparallel \\ 0}}{\sigma_2^2} + \underset{\substack{\shortparallel \\ \rho}}{\sigma_1\sigma_2} + \underset{\substack{\shortparallel \\ (-1)^{n^2}\sigma_1\sigma_2 = (-1)^{n^2}\rho = \rho}}{\sigma_2\sigma_1} = 2\rho$$

$\Rightarrow H(f) = 2$

<u>Prop</u> $H : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is an homomorphism of groups (i.e. $H(f+g) = H(f) + H(g)$)

proof $f+g$

S^{2m-1}



i.e. $f+g: S^{2m-1} \xrightarrow{Q} S^{2m-1} \vee S^{2m-1} \xrightarrow{f \vee g} S^m \vee S^m \xrightarrow{\nabla} S^m$

$$\begin{tikzcd}
S^m \arrow[d, hook] & S^m \vee S^m \arrow[l] \arrow[d, hook] \arrow[r, equal] & S^m \vee S^m \arrow[d, hook] \\
K := S^m \cup_{f+g} e^{2m} & \underbrace{(S^m \vee S^m) \cup_{(f \vee g)Q} e^{2m}}_{=:L} \arrow[l, "p"'] \arrow[r, "q"] & \underbrace{(S^m \cup_f e^{2m}) \vee (S^m \cup_g e^{2m})}_{=:M}
\end{tikzcd}$$

$K = S^m \cup_{f+g} e^{2m}$

$\langle \sigma \rangle = H^m(K)$ $\qquad$ $\langle \tau \rangle = H^{2m}(K)$

$$p^*(\sigma^2) = p^*(H(f+g)\tau) = H(f+g)p^*\tau$$

$$= H(f+g)\rho$$

$\langle \rho \rangle = H^{2m}(L)$

$\sigma_1, \sigma_2 \in H^m(S^m \vee S^m)$ generators

$p^*(\sigma^2) = (\sigma_1 + \sigma_2)^2$ as before for ∇

CLAIM $\quad H(f+g)\rho = (H(f) + H(g))\rho$

$$H(f+g)\rho = H(f+g)p^*\tau = p^*(\sigma^2) = (\sigma_1+\sigma_2)^2$$

$$= H(f)\underbrace{q^*\tau_1}_{=\rho} + H(g)\underbrace{q^*\tau_2}_{=\rho} = (H(f)+H(g))\rho$$

τ_1, τ_2 generators of $H^{2n}(M)$

Corollary (Serre finiteness theorem) particular case

If n is even, $\pi_{2n-1}(S^n)$ contains $\mathbb{Z}$ as direct summand.

proof take f st. $H(f) = 2$

f must have infinite order since otherwise 2 would have finite order in $\mathbb{Z}$

$\Rightarrow \langle f \rangle \cong \mathbb{Z}$

HOPF INVARIANT ONE

$\sigma^2 = H(f)\tau$

<u>Theorem</u> $f : S^{2n-1} \to S^n$ s.t. $H(f) = 1$

$\Rightarrow n = 2^m$ for some m

We will prove this by looking at the mod 2 cohomology of $K := S^n \cup_f e^{2n}$

In fact, $H(f) \equiv 1 \pmod 2 \Rightarrow$

$Sq^n \sigma = \sigma^2 = H(f) \cdot \tau = \tau$ in $H^{2n}(K; \mathbb{Z}_2)$

where $0 < t < n$

$Sq^t : H^n(K; \mathbb{Z}_2) \to H^{n+t}(K; \mathbb{Z}_2) = 0$

We will show that $n \neq 2^m \Rightarrow$

Sq^n is decomposable $\left(Sq^n = \sum_{0<t<n} a_t \overset{0}{\cancel{Sq^t}} \right)$

$\Rightarrow Sq^n = 0$ but this cannot be the case since $Sq^n \sigma = \tau \neq 0$

<u>theorem</u> Sq^i indecomposable $\Rightarrow i = 2^m$

<u>proof</u> if $i \neq 2^m$ we can write

$$i = a + 2^m \qquad 0 < a < 2^m$$
$$= a + b \qquad b := 2^m$$

Adem relations

$$Sq^a Sq^b = \sum_{c \geq 0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c =$$

$$= \binom{b-1}{a} \underbrace{Sq^{a+b}}_{= Sq^i} + \sum_{c>0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

Now we use Lucas theorem

$b = 2^m \qquad b-1 = 2^m - 1 = \sum_{k=0}^{m-1} 2^k \leftarrow$ (a famous sum, easy exercise on mathematical induction)

$$\binom{b-1}{a} \equiv \prod_{k=0}^{m-1} \binom{1}{a_k} \equiv \prod_{k=0}^{m-1} 1 = 1 \pmod 2$$

$\leftarrow \binom{1}{0} = \binom{1}{1} = 1$ & $a \in \{0,1\}$ when we consider the p-adic expansion of a

$b_k = 1 \ \forall\, 0 \leq k \leq m-1$

LUCAS THM

$a = \sum a_i p^i$

$b = \sum b_i p^i$

$$\binom{b}{a} \equiv \prod_i \binom{b_i}{a_i} \pmod p$$

$$Sq^i = Sq^a Sq^b - \sum_{c>0} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c$$

$\Rightarrow Sq^i$ decomposable